Local optimality in algebraic path problems (with help from Coq and Ssreflect)

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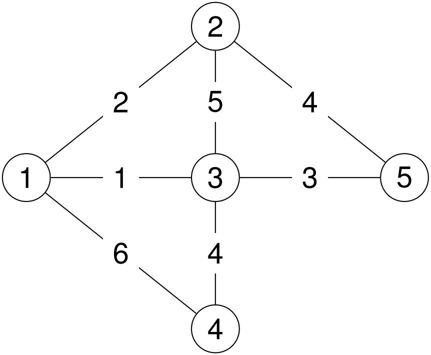
Background

- Internet routing has evolved organically, by the expedient hack....
- ... basic principles need to be uncovered by reverse engineering.
- In the process, a new type of path problem is discovered!
- This may have widespread applicability beyond routing — perhaps in operations research, combinatorics, and other branches of Computer Science.

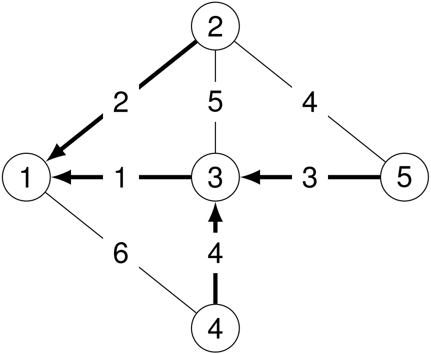
Shortest paths example, $\mathrm{sp} = (\mathbb{N}^{\infty},\ \min,\ +)$



The adjacency matrix

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{array}\right] \end{array}$$

Shortest paths example, continued



Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

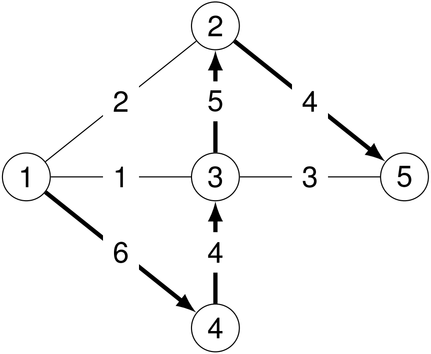
$$\mathbf{A}^* = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

Matrix $\mathbf{A}^*$ solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

Widest paths example, $(\mathbb{N}^{\infty}, \max, \min)$



Bold arrows indicate the widest-path tree rooted at 1.

The routing matrix

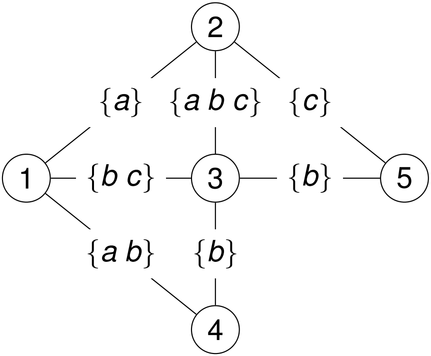
$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{array}\right] \end{array}$$

Matrix $\mathbf{A}^*$ solves this global optimality problem:

$$\mathbf{A}^*(i, j) = \max_{p \in P(i, j)} w(p),$$

where $w(p)$ is now the minimal edge weight in p.

Fun example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$



We want a Matrix $\mathbf{A}^*$ to solve this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \bigcup_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the intersection of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{A}^*(i,\ j)$ to mean that there is at least one path from i to j with x in every arc weight along the path.

Fun example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$

The matrix $\mathbf{A}^*$

$$
\begin{array}{c c}
 & \begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\} & \{a\ b\} & \{a\ b\} & \{a\ b\ c\} & \{b\} \\
\{b\ c\} & \{b\ c\} & \{b\ c\} & \{b\} & \{a\ b\ c\}
\end{bmatrix}
\end{array}
$$

Semirings

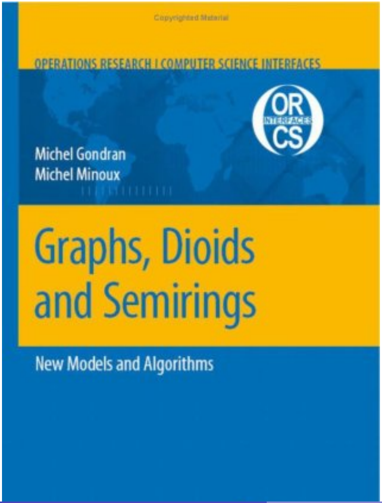
A few examples

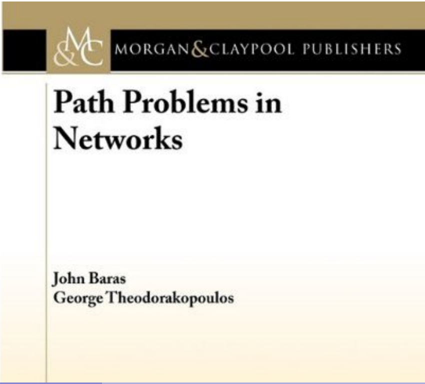
name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	possible routing use
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N}^{\infty}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-path routing
	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes?
	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes?

Path problems focus on global optimality

$$\mathbf{A}^*(i, j) = \bigoplus_{p \in P(i, j)} w(p)$$

Recommended Reading





What algebraic properties are needed for efficient computation of global optimality?

Distributivity

$$\begin{array}{rcrcl} \text{L.D} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c), \\ \text{R.D} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c). \end{array}$$

What is this in $\mathrm{sp} = (\mathbb{N}^{\infty},\ \min,\ +)$?

$$\begin{array}{rcrcl} \text{L.DIST} & : & a + (b \min c) & = & (a + b) \min (a + c), \\ \text{R.DIST} & : & (a \min b) + c & = & (a + c) \min (b + c). \end{array}$$

(I am ignoring all of the other semiring axioms here ...)

Some realistic metrics are not distributive!

Two ways of forming “lexicographic” combination of shortest paths sp and bandwidth bw.

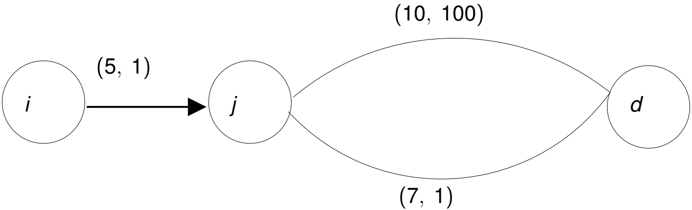
Widest shortest paths

- metric values of form (d, b)
- d in sp
- b in bw
- consider d first, break ties with b
- is distributive (some details ignored ...)

Shortest Widest paths

- metric values of form (b, d)
- d in sp
- b in bw
- consider b first, break ties with d
- not distributive

Example



- node j prefers $(10,\ 100)$ over $(7,\ 1)$.
- node i prefers $(5,\ 2)$ over $(5,\ 101)$.

$$(5,\ 1) \otimes ((10,\ 100) \oplus (7,\ 1)) = (5,\ 1) \otimes (10,\ 100) = (5,\ 101)$$
$$((5,\ 1) \otimes (10,\ 101)) \oplus ((5,\ 1) \otimes (7,\ 1)) = (5,\ 101) \oplus (5,\ 2) = (5,\ 2)$$

Left-Local Optimality

Say that $\mathbf{L}$ is a left locally-optimal solution when

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{L}(i,\ j) = \bigoplus_{q \in V} \mathbf{A}(i,\ q) \otimes \mathbf{L}(q,\ j)$$

- $\mathbf{L}(i,\ j)$ is the best possible value given the values $\mathbf{L}(q,\ j)$, for all out-neighbors q of source i.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees <u>from</u>** i (think Bellman-Ford).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees <u>to</u>** i.
- Works well with hop-by-hop forwarding from i.

Right-Local Optimality

Say that **R** is a right locally-optimal solution when

$$\mathbf{R} = (\mathbf{R} \otimes \mathbf{A}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{R}(i,\ j) = \bigoplus_{q \in V} \mathbf{R}(i,\ q) \otimes \mathbf{A}(q,\ j)$$

- $\mathbf{R}(i,\ j)$ is the best possible value given the values $\mathbf{R}(q,\ j)$, for all in-neighbors q of destination j.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees from** i (think Dijkstra).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees to** i.
- Does not work well with hop-by-hop forwarding from i.

With and Without Distributivity

With

For semirings, the three optimality problems are essentially the same — locally optimal solutions are globally optimal solutions.

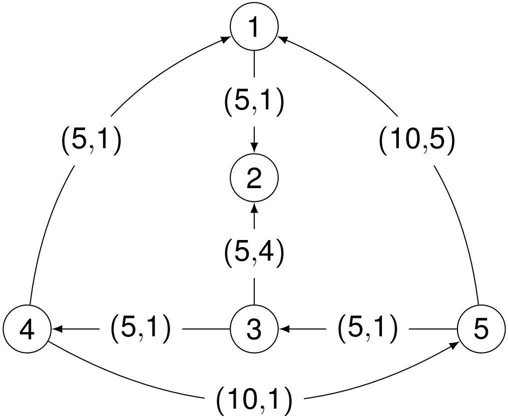
$$\mathbf{A}^* = \mathbf{L} = \mathbf{R}$$

Without

Suppose that we drop distributivity and $\mathbf{A}^*$, $\mathbf{L}$, $\mathbf{R}$ exist. It may be the case they they are all distinct.

Health warning : matrix multiplication over structures lacking distributivity is not associative!

Example



(bandwidth, distance) with lexicographic order (bandwidth first).

Global optima

$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\ \begin{bmatrix} (\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) & (0, \infty) \\ (0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) & (0, \infty) \\ (5, 2) & (5, 3) & (\infty, \ 0) & (5, 1) & (5, 2) \\ (10, 6) & (5, 2) & (5, 2) & (\infty, 0) & (10, 1) \\ (10, 5) & (5, 4) & (5, 1) & (5, 2) & (\infty, 0) \end{bmatrix} \end{array},$$

Left local optima

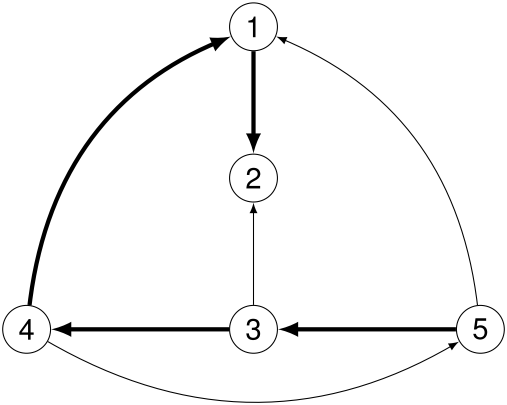
$$
\mathbf{L} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\
\begin{bmatrix}
(\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) & (0, \infty) \\
(0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) & (0, \infty) \\
(\mathbf{5}, \mathbf{7}) & (5, 3) & (\infty, 0) & (5, 1) & (5, 2) \\
(10, 6) & (5, 2) & (5, 2) & (\infty, 0) & (10, 1) \\
(10, 5) & (5, 4) & (5, 1) & (5, 2) & (\infty, 0)
\end{bmatrix}
\end{array},
$$

Entries marked in **bold** indicate those values which are not globally optimal.

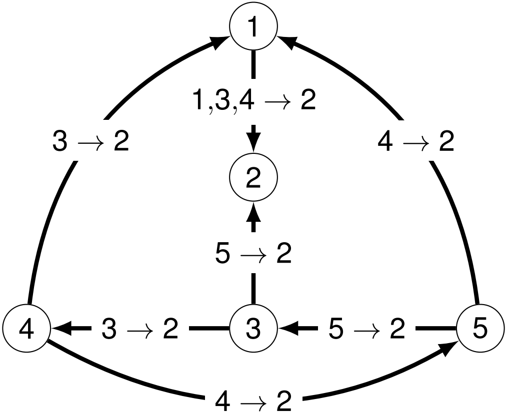
Right local optima

$$
\mathbf{R} = \begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
(\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) & (0, \infty) \\
(0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) & (0, \infty) \\
(5, 2) & (5, 3) & (\infty,\ 0) & (5,\ 1) & (5, 2) \\
(10, 6) & (\mathbf{5}, \mathbf{6}) & (5, 2) & (\infty, 0) & (10, 1) \\
(10, 5) & (\mathbf{5}, \mathbf{5}) & (5, 1) & (5, 2) & (\infty, 0)
\end{array}\right]
\end{array},
$$

Left-locally optimal paths to node 2



Right-locally optimal paths to node 2



Bellman-Ford can compute left-local solutions

$$\mathbf{A}^{[0]} = \mathbf{I}$$
$$\mathbf{A}^{[k+1]} = (\mathbf{A} \otimes \mathbf{A}^{k}) \oplus \mathbf{I},$$

- Bellman-ford algorithm must be modified to ensure only loop-free paths are inspected.
- $(S, \oplus, \overline{0})$ is a commutative, idempotent, and selective monoid,
- $(S, \otimes, \overline{1})$ is a monoid,
- $\overline{0}$ is the annihilator for $\otimes$,
- $\overline{1}$ is the annihilator for $\oplus$,
- Left strictly inflationarity, L.S.INF : $\forall a, b : a \neq \overline{0} \implies a < a \otimes b$
- Here $a \leq b \equiv a = a \oplus b$.

Convergence to a unique left-local solution is guaranteed. Currently no polynomial bound is known on the number of iterations required.

Dijkstra's algorithm can work for right-local optima!

Input : adjacency matrix $\mathbf{A}$ and source vertex $i \in V$,
Output : the i-th row of $\mathbf{R}$, $\mathbf{R}(i, \_)$.

begin
  $S \leftarrow \{i\}$
  $\mathbf{R}(i, i) \leftarrow \overline{1}$
  for each $q \in V - \{i\}$: $\mathbf{R}(i, q) \leftarrow \mathbf{A}(i, q)$
  while $S \neq V$
    begin
      find $q \in V - S$ such that $\mathbf{R}(i, q)$ is $\leq^{L}_{\oplus}$ -minimal
      $S \leftarrow S \cup \{q\}$
      for each $j \in V - S$
        $\mathbf{R}(i, j) \leftarrow \mathbf{R}(i, j) \oplus (\mathbf{R}(i, q) \otimes \mathbf{A}(q, j))$
    end
end

From right to left ...

Need left-local optima!

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I} \quad \Longleftrightarrow \quad \mathbf{L}^T = (\mathbf{L}^T \hat{\otimes}^T \mathbf{A}^T) \oplus \mathbf{I}$$

where $\otimes^T$ is matrix multiplication defined with as

$$a \otimes^T b = b \otimes a$$

and we assume left-inflationarity holds, $\text{L.INF} : \forall a, b : a \leq b \otimes a$.

Each node would have to solve the entire “all pairs” problem.

Minimal subset of semiring axioms needed right-local Dijkstra

~~Semiring~~ Axioms

$$
\begin{array}{rcrcl}
\text{ADD.ASSOCIATIVE} & : & a \oplus (b \oplus c) & = & (a \oplus b) \oplus c \\
\text{ADD.COMMUTATIVE} & : & a \oplus b & = & b \oplus a \\
\text{ADD.LEFT.ID} & : & \overline{0} \oplus a & = & a \\
\cancel{\text{MULT.ASSOCIATIVE}} & : & \cancel{a \otimes (b \otimes c)} & \cancel{=} & \cancel{(a \otimes b) \otimes c} \\
\text{MULT.LEFT.ID} & : & \overline{1} \otimes a & = & a \\
\cancel{\text{MULT.RIGHT.ID}} & : & \cancel{a \otimes \overline{1}} & \cancel{=} & \cancel{a} \\
\cancel{\text{MULT.LEFT.ANN}} & : & \cancel{\overline{0} \otimes a} & \cancel{=} & \cancel{\overline{0}} \\
\cancel{\text{MULT.RIGHT.ANN}} & : & \cancel{a \otimes \overline{0}} & \cancel{=} & \cancel{\overline{0}} \\
\cancel{\text{L.DISTRIBUTIVE}} & : & \cancel{a \otimes (b \oplus c)} & \cancel{=} & \cancel{(a \otimes b) \oplus (a \otimes c)} \\
\cancel{\text{R.DISTRIBUTIVE}} & : & \cancel{(a \oplus b) \otimes c} & \cancel{=} & \cancel{(a \otimes c) \oplus (b \otimes c)}
\end{array}
$$

Additional axioms needed right-local Dijkstra

$$\begin{array}{rcrcl}
\text{ADD.SELECTIVE} & : & a \oplus b & \in & \{a,\ b\} \\
\text{ADD.LEFT.ANN} & : & \overline{1} \oplus a & = & \overline{1} \\
\text{ADD.RIGHT.ANN} & : & a \oplus \overline{1} & = & \overline{1} \\
\text{RIGHT.ABSORBTION} & : & a \oplus (a \otimes b) & = & a
\end{array}$$

RIGHT.ABSORBTION gives inflationarity, $\forall a, b : a \leq a \otimes b$.

The goal

Given adjacency matrix $\mathbf{A}$ and source vertex $i \in V$, Dijkstra's algorithm will compute $\mathbf{R}(i, \_)$ such that

$$\forall j \in V : \mathbf{R}(i, j) = \mathbf{I}(i,j) \oplus \bigoplus_{q \in V} \mathbf{R}(i, q) \otimes \mathbf{A}(q, j).$$

Main invariant

$$\forall k : 1 \leq k \leq \mid V \mid \implies \forall j \in S_k : \mathbf{R}_k(i, j) = \mathbf{I}(i,j) \oplus \bigoplus_{q \in S_k} \mathbf{R}_k(i, q) \otimes \mathbf{A}(q, j)$$

Routing in Equilibrium. João Luís Sobrinho and Timothy G. Griffin. The 19th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2010).

A small snapshot using Coq + ssreflect

```
Variable plus_associative      : ∀x y z, x ⊕ (y ⊕ z) = (x ⊕ y) ⊕ z.
Variable plus_commutative      : ∀x y, x ⊕ y = y ⊕ x.
Variable plus_selective        : ∀x y, (x ⊕ y == x) || (x ⊕ y == y).

(* identities                                                      *)
Variable zero_is_left_plus_id  : ∀x, zero ⊕ x = x.
Variable one_is_left_times_id  : ∀x, one ⊗ x = x.

(* one is additive annihilator                                      *)
Variable one_is_left_plus_ann  : ∀x, one ⊕ x = one.
Variable one_is_right_plus_ann : ∀x, x ⊕ one = one.

(* right absorbtion                                                 *)
Variable right_absorption      : ∀ a b : T, a ⊕ (a ⊗ b) == a.

Definition lno (a b : T) := a ⊕ b == a.
Notation "A ≦ B" := (lno A B) (at level 60).

Lemma lno_right_increasing : ∀ a b : T,  a ≦ a ⊗ b.
```

Using Coq + Ssreflect

Talk will finish with an interactive look at a proof

http://www.cl.cam.ac.uk/ tgg22/metarouting/rie-1.0.v