Local optimality in algebraic path problems 
(with help from Coq and Ssreflect)

Timothy G. Griffin

Computer Laboratory
University of Cambridge, UK
timothy.griffin@cl.cam.ac.uk

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Internet routing has evolved organically, by the expedient hack....
... basic principles need to be uncovered by reverse engineering.
In the process, a new type of path problem is discovered!
This may have widespread applicability beyond routing — perhaps in operations research, combinatorics, and other branches of Computer Science.
Shortest paths example, $sp = (\mathbb{N}^\infty, \min, +)$

The adjacency matrix

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & \infty & 2 & 1 & 6 & \infty \\
2 & 2 & \infty & 5 & \infty & 4 \\
3 & 1 & 5 & \infty & 4 & 3 \\
4 & 6 & \infty & 4 & \infty & \infty \\
5 & \infty & 4 & 3 & \infty & \infty
\end{bmatrix}$$
Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

\[
A^* = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 2 & 1 & 5 & 4 \\
2 & 2 & 0 & 3 & 7 & 4 \\
3 & 1 & 3 & 0 & 4 & 3 \\
4 & 5 & 7 & 4 & 0 & 7 \\
5 & 4 & 4 & 3 & 7 & 0 \\
\end{bmatrix}
\]

Matrix \(A^*\) solves this global optimality problem:

\[
A^*(i, j) = \min_{p \in P(i, j)} w(p),
\]

where \(P(i, j)\) is the set of all paths from \(i\) to \(j\).
Widest paths example, \((\mathbb{N}^\infty, \max, \min)\)

The routing matrix

\[
A^* = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & \infty & 4 & 4 & 6 & 4 \\
2 & 4 & \infty & 5 & 4 & 4 \\
3 & 4 & 5 & \infty & 4 & 4 \\
4 & 6 & 4 & 4 & \infty & 4 \\
5 & 4 & 4 & 4 & 4 & \infty \\
\end{bmatrix}
\]

Matrix \(A^*\) solves this global optimality problem:

\[
A^*(i, j) = \max_{p \in P(i, j)} w(p),
\]

where \(w(p)\) is now the minimal edge weight in \(p\).
Fun example, \((2\{a, b, c\}, \cup, \cap)\)

We want a Matrix \(A^*\) to solve this global optimality problem:

\[
A^*(i, j) = \bigcup_{p \in P(i, j)} w(p),
\]

where \(w(p)\) is now the intersection of all edge weights in \(p\).

For \(x \in \{a, b, c\}\), interpret \(x \in A^*(i, j)\) to mean that there is at least one path from \(i\) to \(j\) with \(x\) in every arc weight along the path.
Fun example, \((2\{a, b, c\}, \cup, \cap)\)

The matrix \(A^*\)

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & \{a, b, c\} & \{a, b, c\} & \{a, b\} & \{b, c\} \\
2 & \{a, b, c\} & \{a, b, c\} & \{a, b\} & \{b, c\} \\
3 & \{a, b, c\} & \{a, b, c\} & \{a, b\} & \{b, c\} \\
4 & \{a, b\} & \{a, b\} & \{a, b\} & \{a, b, c\} & \{b\} \\
5 & \{b, c\} & \{b, c\} & \{b, c\} & \{b\} & \{a, b, c\}
\end{pmatrix}
\]
## Semirings

### A few examples

<table>
<thead>
<tr>
<th>name</th>
<th>$S$</th>
<th>$\oplus$,</th>
<th>$\otimes$</th>
<th>$\bar{0}$</th>
<th>$\bar{1}$</th>
<th>possible routing use</th>
</tr>
</thead>
<tbody>
<tr>
<td>sp</td>
<td>$\mathbb{N}^\infty$</td>
<td>$\min$</td>
<td>$+$</td>
<td>$\infty$</td>
<td>$0$</td>
<td>minimum-weight routing</td>
</tr>
<tr>
<td>bw</td>
<td>$\mathbb{N}^\infty$</td>
<td>$\max$</td>
<td>$\min$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>greatest-capacity routing</td>
</tr>
<tr>
<td>rel</td>
<td>$[0, 1]$</td>
<td>$\max$</td>
<td>$\times$</td>
<td>$0$</td>
<td>$1$</td>
<td>most-reliable routing</td>
</tr>
<tr>
<td>use</td>
<td>${0, 1}$</td>
<td>$\max$</td>
<td>$\min$</td>
<td>$0$</td>
<td>$1$</td>
<td>usable-path routing</td>
</tr>
<tr>
<td></td>
<td>$2^W$</td>
<td>$\cup$</td>
<td>$\cap$</td>
<td>${}$</td>
<td>$W$</td>
<td>shared link attributes?</td>
</tr>
<tr>
<td></td>
<td>$2^W$</td>
<td>$\cap$</td>
<td>$\cup$</td>
<td>${}$</td>
<td>$W$</td>
<td>shared path attributes?</td>
</tr>
</tbody>
</table>

### Path problems focus on global optimality

\[
A^*(i, j) = \bigoplus_{p \in P(i, j)} w(p)
\]
Recommended Reading

- *Graphs, Dioids and Semirings: New Models and Algorithms* by Michel Gondran and Michel Minoux
  - Morgan & Claypool Publishers

- *Path Problems in Networks* by John Baras and George Theodorakopoulos
What algebraic properties are needed for efficient computation of global optimality?

**Distributivity**

- **L.D**: \( a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \),
- **R.D**: \((a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)\).

What is this in \( sp = (\mathbb{N}^\infty, \min, +) \)?

- **L.DIST**: \( a + (b \min c) = (a + b) \min (a + c) \),
- **R.DIST**: \((a \min b) + c = (a + c) \min (b + c)\).

(I am ignoring all of the other semiring axioms here ...)
Some realistic metrics are not distributive!

Two ways of forming “lexicographic” combination of shortest paths $sp$ and bandwidth $bw$.

**Widest shortest paths**
- metric values of form $(d, b)$
- $d$ in $sp$
- $b$ in $bw$
- consider $d$ first, break ties with $b$
- is distributive (some details ignored ...)

**Shortest Widest paths**
- metric values of form $(b, d)$
- $d$ in $sp$
- $b$ in $bw$
- consider $b$ first, break ties with $d$
- not distributive
Example

- node $j$ prefers $(10, 100)$ over $(7, 1)$.
- node $i$ prefers $(5, 2)$ over $(5, 101)$.

\[
(5, 1) \otimes ((10, 100) \oplus (7, 1)) = (5, 1) \otimes (10, 100) = (5, 101) \\
((5, 1) \otimes (10, 101)) \oplus ((5, 1) \otimes (7, 1)) = (5, 101) \oplus (5, 2) = (5, 2)
\]
Left-Local Optimality

Say that $L$ is a left locally-optimal solution when

$$L = (A \otimes L) \oplus I.$$ 

That is, for $i \neq j$ we have

$$L(i, j) = \bigoplus_{q \in V} A(i, q) \otimes L(q, j).$$

- $L(i, j)$ is the best possible value given the values $L(q, j)$, for all out-neighbors $q$ of source $i$.
- Rows $L(i, \_)$ represents out-trees from $i$ (think Bellman-Ford).
- Columns $L(\_, i)$ represents in-trees to $i$.
- Works well with hop-by-hop forwarding from $i$. 
Right-Local Optimality

Say that $R$ is a right locally-optimal solution when

$$R = (R \otimes A) \oplus I.$$

That is, for $i \neq j$ we have

$$R(i, j) = \bigoplus_{q \in V} R(i, q) \otimes A(q, j)$$

- $R(i, j)$ is the best possible value given the values $R(q, j)$, for all in-neighbors $q$ of destination $j$.
- Rows $L(i, \_)$ represents **out-trees from** $i$ (think Dijkstra).
- Columns $L(\_, i)$ represents **in-trees to** $i$.
- Does not work well with hop-by-hop forwarding from $i$. 
With and Without Distributivity

With

For semirings, the three optimality problems are essentially the same — locally optimal solutions are globally optimal solutions.

\[ A^* = L = R \]

Without

Suppose that we drop distributivity and \( A^*, L, R \) exist. It may be the case they they are all distinct.

Health warning: matrix multiplication over structures lacking distributivity is not associative!
Example

(bandwidth, distance) with lexicographic order (bandwidth first).
Global optima

\[
A^* = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & (\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) \\
2 & (0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) \\
3 & (5, 2) & (5, 3) & (\infty, 0) & (5, 1) & (5, 2) \\
4 & (10, 6) & (5, 2) & (5, 2) & (\infty, 0) & (10, 1) \\
5 & (10, 5) & (5, 4) & (5, 1) & (5, 2) & (\infty, 0)
\end{bmatrix},
\]
Left local optima

\[ L = \begin{bmatrix}
    (\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) & (0, \infty) \\
    (0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) & (0, \infty) \\
    (5, 7) & (5, 3) & (\infty, 0) & (5, 1) & (5, 2) \\
    (10, 6) & (5, 2) & (5, 2) & (\infty, 0) & (10, 1) \\
    (10, 5) & (5, 4) & (5, 1) & (5, 2) & (\infty, 0)
\end{bmatrix}, \]

Entries marked in **bold** indicate those values which are not globally optimal.
Right local optima

\[ R = \begin{bmatrix}
  1 & (\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) & (0, \infty) \\
  2 & (0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) & (0, \infty) \\
  3 & (5, 2) & (5, 3) & (\infty, 0) & (5, 1) & (5, 2) \\
  4 & (10, 6) & (5, 6) & (5, 2) & (\infty, 0) & (10, 1) \\
  5 & (10, 5) & (5, 5) & (5, 1) & (5, 2) & (\infty, 0)
\end{bmatrix}, \]
Left-locally optimal paths to node 2
Right-locally optimal paths to node 2
Bellman-Ford can compute left-local solutions

\[
A^{[0]} = I \\
A^{[k+1]} = (A \otimes A^k) \oplus I,
\]

- Bellman-ford algorithm must be modified to ensure only loop-free paths are inspected.
- \((S, \oplus, 0)\) is a commutative, idempotent, and selective monoid,
- \((S, \otimes, 1)\) is a monoid,
- \(\overline{0}\) is the annihilator for \(\otimes\),
- \(\overline{1}\) is the annihilator for \(\oplus\),
- Left strictly inflationarity, \(\text{L.S.INF} : \forall a, b : a \neq 0 \implies a < a \otimes b\)
- Here \(a \leq b \equiv a = a \oplus b\).

Convergence to a unique left-local solution is guaranteed. Currently no polynomial bound is known on the number of iterations required.
Dijkstra’s algorithm can work for right-local optima!

**Input**: adjacency matrix $A$ and source vertex $i \in V$,  
**Output**: the $i$-th row of $R$, $R(i, \_)$.

```
begin
S ← \{i\}
R(i, i) ← 1
for each $q \in V - \{i\}$ : $R(i, q) ← A(i, q)$
while $S \neq V$
    begin
        find $q \in V - S$ such that $R(i, q)$ is $\leq^L$-minimal
        $S ← S \cup \{q\}$
        for each $j \in V - S$
            $R(i, j) ← R(i, j) \oplus (R(i, q) \otimes A(q, j))$
    end
end
```
Need left-local optima!

\[ L = (A \otimes L) \oplus I \iff L^T = (L^T \otimes^T A^T) \oplus I \]

where \( \otimes^T \) is matrix multiplication defined with as

\[ a \otimes^T b = b \otimes a \]

and we assume left-inflationarity holds, \( L.\ INF : \forall a, b : a \leq b \otimes a. \)

Each node would have to solve the entire "all pairs" problem.
Minimal subset of semiring axioms needed right-local Dijkstra

**Semiring Axioms**

<table>
<thead>
<tr>
<th>Axiom Description</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ADD. ASSOCIATIVE</strong></td>
<td>$a \oplus (b \oplus c) = (a \oplus b) \oplus c$</td>
</tr>
<tr>
<td><strong>ADD. COMMUTATIVE</strong></td>
<td>$a \oplus b = b \oplus a$</td>
</tr>
<tr>
<td><strong>ADD. LEFT. ID</strong></td>
<td>$0 \oplus a = a$</td>
</tr>
<tr>
<td><strong>MULT. ASSOCIATIVE</strong></td>
<td>$a \otimes (b \otimes c) = (a \otimes b) \otimes c$</td>
</tr>
<tr>
<td><strong>MULT. LEFT. ID</strong></td>
<td>$1 \otimes a = a$</td>
</tr>
<tr>
<td><strong>MULT. RIGHT. ID</strong></td>
<td>$a /i/ 1 = a$</td>
</tr>
<tr>
<td><strong>MULT. LEFT. ANN</strong></td>
<td>$0 /i/ a = 0$</td>
</tr>
<tr>
<td><strong>MULT. RIGHT. ANN</strong></td>
<td>$a /i/ 0 = 0$</td>
</tr>
<tr>
<td><strong>L. DISTRIBUTIVE</strong></td>
<td>$a \otimes (b \oplus c) \uplus (a \otimes b) \oplus (a \otimes c)$</td>
</tr>
<tr>
<td><strong>R. DISTRIBUTIVE</strong></td>
<td>$(a \oplus b) \otimes c \uplus (a \otimes c) \otimes (b \otimes c)$</td>
</tr>
</tbody>
</table>
Additional axioms needed right-local Dijkstra

\[
\begin{align*}
\text{ADD.SELECTIVE} & : \quad a \oplus b \in \{a, b\} \\
\text{ADD.LEFT.ANN} & : \quad \bar{1} \oplus a = \bar{1} \\
\text{ADD.RIGHT.ANN} & : \quad a \oplus \bar{1} = \bar{1} \\
\text{RIGHT.ABSORBTION} & : \quad a \oplus (a \otimes b) = a
\end{align*}
\]

RIGHT.ABSORBTION gives inflationarity, \( \forall a, b : a \leq a \otimes b. \)
The goal

Given adjacency matrix $A$ and source vertex $i \in V$, Dijkstra’s algorithm will compute $R(i, \_)$ such that

$$\forall j \in V : R(i, j) = I(i, j) \oplus \bigoplus_{q \in V} R(i, q) \otimes A(q, j).$$

Main invariant

$$\forall k : 1 \leq k \leq |V| \implies \forall j \in S_k : R_k(i, j) = I(i, j) \oplus \bigoplus_{q \in S_k} R_k(i, q) \otimes A(q, j)$$

A small snapshot using Coq + ssreflect

Variable plus_associative : \forall x y z, x \oplus (y \oplus z) = (x \oplus y) \oplus z.
Variable plus_commutative : \forall x y, x \oplus y = y \oplus x.
Variable plus_selective : \forall x y, (x \oplus y == x) || (x \oplus y == y).

(* identities *)
Variable zero_is_left_plus_id : \forall x, zero \oplus x = x.
Variable one_is_left_times_id : \forall x, one \odot x = x.

(* one is additive annihilator *)
Variable one_is_left_plus_ann : \forall x, one \oplus x = one.
Variable one_is_right_plus_ann : \forall x, x \oplus one = one.

(* right absorption *)
Variable right_absorption : \forall a b : T, a \oplus (a \odot b) == a.

Definition lno (a b : T) := a \oplus b == a.
Notation "A \leq B" := (lno A B) (at level 60).

Lemma lno_right_increasing : \forall a b : T, a \leq a \oplus b.
Using Coq + Ssreflect

Talk will finish with an interactive look at a proof

http://www.cl.cam.ac.uk/ tgg22/metarouting/rie-1.0.v