Entropy of Random Geometric Graphs

Justin P. Coon

with thanks to N. Warsi, O. Georgiou and C. P. Dettmann

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Random Geometric Graphs and Spatially Embedded Networks

The model...

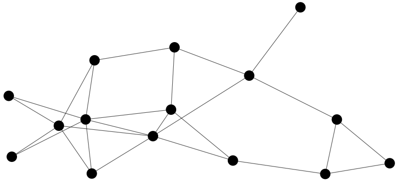
- Node positions are random
- Existence of an edge (i, j) depends on distance between nodes i and j
- Notion of distance implies embedding in some space (e.g., Euclidean)

Applications...

- Human/animal interactions
- Nano processes (e.g., carbon nanotubes on a polymer substrate)
- Wireless communication networks

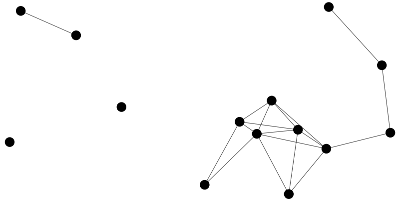
Erdős-Rényi Graph vs Random Geometric Graph

ER Graphs



- Edge probability $\wp$
- Independent of embedding

RG Graphs



- Edge probability $\wp(s_{i,j})$
- Distribution of $s_{i,j}$ important

Graph Entropy

Consider a graph $G = (\mathcal{V}, \mathcal{E})$ to be a set of vertices (nodes) and a set of edges (links).

- $\binom{n}{2} = n(n-1)/2$ possible edge configurations
- $2^{n(n-1)/2}$ possible graphs without considering node locations

The entropy of G is a measure of disorder or the amount of information contained in the graph distribution.

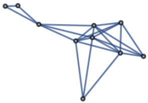
$$H(G) = \mathbb{E}[-\log(\mathbb{P}(G))]$$

** Logarithms are base e.*

ER Graph Entropy

For an ER graph, all edges occur independently with probability $\wp$. To calculate entropy...

- Map G to $n(n-1)/2$ Bernoulli variables $X_1, \ldots, X_{n(n-1)/2}$



$\mapsto$ **110 100 010 . . .**

- Independence of $\{X_i\}$ implies

$$H(G) = H(X_1, \ldots, X_{n(n-1)/2}) = \sum_i H(X_i) = \binom{n}{2} H(\wp)$$

- Entropy of single edge is

$$H(\wp) = -\wp \log \wp - (1-\wp)\log(1-\wp)$$

ER Graph Entropy

Examples

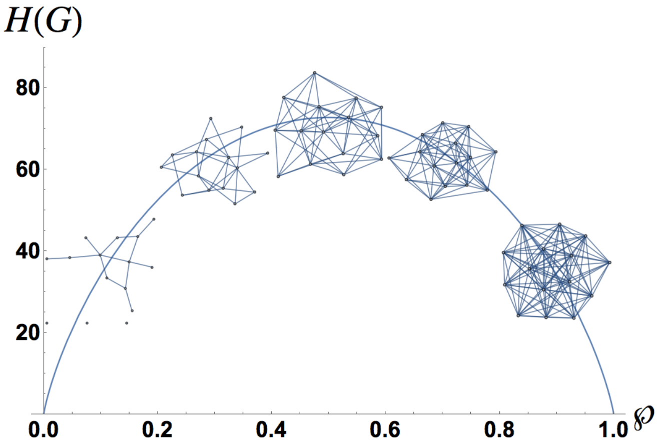


Figure : Entropy of ER graph with $n = 15$ nodes.

RG Graph Entropy

For an RG graph, edge (i, j) occurs with probability $\wp_{i,j}$, which depends on the distance $s_{i,j} = \|\mathbf{r}_i - \mathbf{r}_j\|$. Consider *conditional entropy*...

$$
\begin{aligned}
H(G|S) &= \mathbb{E}[H(G|s_{1,2}, \ldots, s_{n-1,n})] \\
&\leq \sum_{i<j} \mathbb{E}[H(X_{i,j}|s_{i,j})] && \text{(independence bound)} \\
&= \binom{n}{2} \mathbb{E}[H(X|s)] \\
&\leq \binom{n}{2} H(\overline{\wp}) && \text{(Jensen's inequality)}
\end{aligned}
$$

where

$$\overline{\wp} := \mathbb{E}[\wp(s_{i,j})].$$

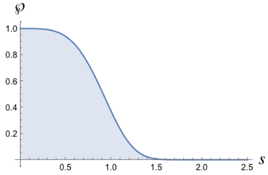
Pair Connection Function

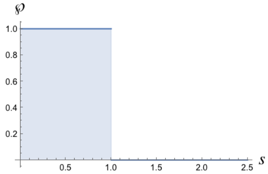
Interesting properties arise from the **distribution of the pair distance** and the **pair connection function**. Let...

$$\wp(s) = \exp(-(s/s_0)^\eta)$$

- Models Rayleigh fading in wireless networks
- Versatility offered by parameter η: probabilistic to deterministic

$$\Rightarrow \overline{\wp} = \int_0^D f(s) \exp(-(s/s_0)^\eta)\,\mathrm{d}s$$





Uniform Node Distributions in a Compact Domain

In d dimensions, the **set covariance** of a convex set $\mathcal{K}$ is given by

$$c_{\mathcal{K}}(\mathbf{s}) = \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{K}}(\mathbf{x}) \mathbf{1}_{\mathcal{K}}(\mathbf{x} - \mathbf{s}) \, \mathrm{d}\mathbf{x}, \qquad \mathbf{s} \in \mathbb{R}^d$$

where $\mathbf{1}_{\mathcal{K}}(\mathbf{x})$ is the indicator function for $\mathbf{x} \in \mathcal{K}$.

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The **isotropised set covariance** is given by

$$\overline{c}_{\mathcal{K}}(s) = \int_{\mathcal{S}^{d-1}} c_{\mathcal{K}}(s\mathbf{u}) \, \mathrm{d}\mathbf{u}, \qquad s \geq 0$$

where $\mathbf{u}$ is a vector denoting a point on the unit sphere $\mathcal{S}^{d-1}$.

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The **pair distance** probability density function is given by

$$f(s) = \frac{2\pi^{d/2} s^{d-1} \overline{c}_{\mathcal{K}}(s)}{\Gamma(d/2)\,\mathrm{vol}(\mathcal{K})^2}.$$

Entropy vs Connection Range s_0

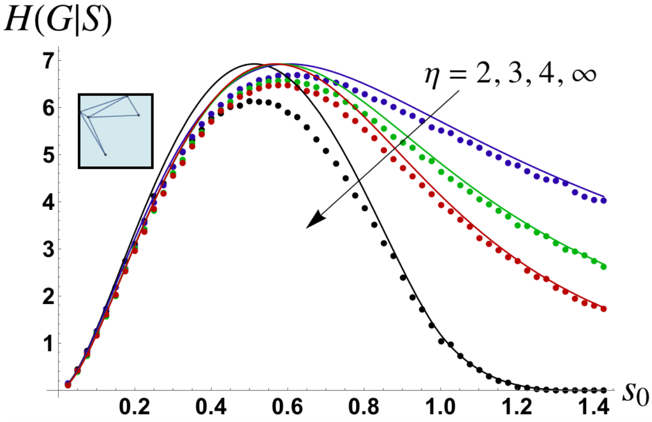
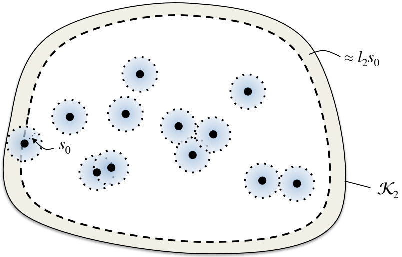


Figure : Entropy of RG graph with $n = 5$ nodes and pair connection functions with typical range s_0 for a unit square. Solid line: upper bound. Markers: numerical simulations.

Uniform Node Distributions in a 2D Compact Domain

Small Typical Connection Range

Consider a compact domain $\mathcal{K}_2$ with area a_2, boundary length l_2 and small typical connection range $s_0 \ll \pi a_2 / l_2 \ll D$.



- Pair distance distribution

$$f(s) = \frac{2\pi s}{a_2}\left(1 - \frac{l_2 s}{\pi a_2}\right)$$

- Soft pair connection probability

$$\overline{\wp} = \frac{2\pi s_0^2 \Gamma(2/\eta)}{\eta a_2}\left(1 - \frac{l_2 s_0}{\pi a_2}\frac{\Gamma(3/\eta)}{\Gamma(2/\eta)}\right)$$

- Hard pair connection probability

$$\overline{\wp} = \frac{\pi s_0^2}{a_2}\left(1 - \frac{2 l_2 s_0}{3\pi a_2}\right)$$

Entropy for Small s_0

Let $u_2 = 2\pi\Gamma(2/\eta)/(\eta a_2)$ denote the *fractional area of a unit soft disc*. For $s_0 \to 0$, we have

$$H(G|S) \leq \binom{n}{2} \left(2u_2 \log\left(\frac{1}{s_0}\right) s_0^2 + u_2(1 - \log u_2)s_0^2 + O(s_0^3 \log s_0) \right)$$

- For fixed s_0, the bound increases with n.
- For fixed n, the bound decreases with s_0 (toward a completely disconnected graph).

Entropy for Small s_0

What about $s_0 = g(n)$?

Fact

For typical connection distances that decay according to the relation

$$s_0^2 \log\left(\frac{1}{s_0}\right) = o\left(\frac{1}{n^2}\right)$$

the entropy $H(G|S) \to 0$ *as* $n \to \infty$.

An Entropy Limit for Small s_0

Fact

The upper bound on the entropy of a graph in $\mathcal{K}_2$ will tend to a limit $\nu > 0$ as $n \to \infty$ if

$$s_0(n) = \exp\left(\frac{1}{2} W_m\left(-\frac{2\nu}{u_2 n^2}\right)\right) = \frac{\sqrt{\nu}}{n\sqrt{u_2 \log n}}\left(1 + O\left(\frac{1}{\log n}\right)\right)$$

where $W_m(x)$ is the lower branch ($-1/e \leq x < 0$ and $W_m \leq -1$) of the solution to $x = W \exp W$.

Entropy vs Number of Nodes n

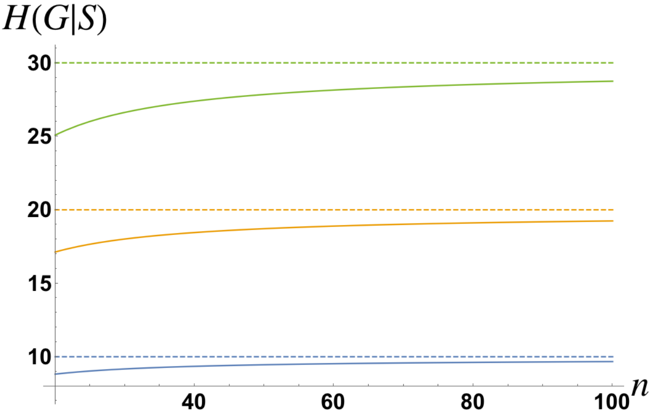


Figure : Upper bound on the entropy of an RG graph in a unit square with $s_0 = \exp\left(\frac{1}{2} W_m \left(-\frac{2\nu}{u_2 n^2}\right)\right)$. Solid line: theory. Dashed line: limit.

Connectivity for Small s_0

Fact

The probability that a graph in $\mathcal{K}_2$ is completely disconnected is well approximated by

$$P = (1 - \overline{\wp})^{n(n-1)/2}.$$

Thus, for $s_0 \to 0$ and $n \to \infty$, a graph is almost surely completely disconnected if

$$s_0 = o\left(\frac{1}{n}\right).$$

The probability of a completely disconnected graph will tend to a limit $\varphi \in (0, 1)$ if

$$s_0(n) = \frac{1}{n}\sqrt{\frac{2}{u_2}\log\left(\frac{1}{\varphi}\right)}$$

Connectivity vs Entropy for Small s_0

Interesting Fact

Letting s_0 tend to zero such that the upper bound on $H(G|S)$ tends to $\nu > 0$ yields

$$P \sim 1 - \frac{\nu}{2 \log n}.$$

I.e., typical connection distances that yield a positive limit on the entropy bound result in almost surely completely disconnected graphs (albeit with a slow convergence).

Arbitrary Node Configurations for $s_0 \gg D$

For the **hard disc model**, the graph G is complete, so $H(G|S) = H(G) = 0$.

For the **soft model** (i.e., $\eta < \infty$)

$$\overline{\wp} = 1 - \frac{\mathbb{E}[s^\eta]}{s_0^\eta} + O\left(\frac{1}{s_0^{2\eta}}\right)$$

provided the ηth, 2ηth, ... moments exist. The conditional entropy is thus bounded by

$$H(G|S) \leq \binom{n}{2} \frac{\eta \mathbb{E}[s^\eta] \log s_0}{s_0^\eta} + O\left(\frac{1}{s_0^\eta}\right), \qquad \eta < \infty$$

Room for Improvement...

- Consider $d > 2$; all small s_0 results generalise
- More to consider for $s_0 \gg D$
- Exact calculations (difficult)
- Strengthen the bound
- Consider other pair connection functions
- Explicit calculations for specific bounding geometries
- Maximum entropy points
- Structural entropy
- Applications (e.g., routing/forwarding tables, protocol design)