Performance of a link in a field of vehicular interferers with hardcore headway distance

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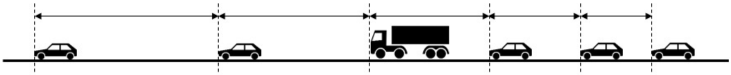
Motivation

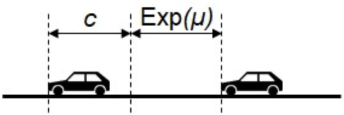
- Headway distance: Distance from the tip of a vehicle to the tip of its follower
- Location of vehicles is commonly modeled by a Poisson point process (PPP) in the literature
- In roads with few number of lanes the PPP assumption might not be accurate as it allows unrealistically small headways



Motivation

Complex headway models in transportation research

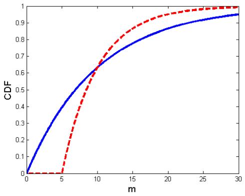
Balancing accuracy and tractability: Cowan M2 headway distance model has two components: A constant tracking distance + a random component following the exponential distribution [Cowan1975]



Intensity of vehicles

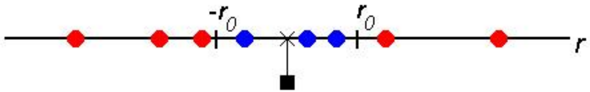
$$\lambda = \frac{\mu}{1+\mu c}$$

Distribution of inter-vehicle distances



System model

- Vehicles – impenetrable disks of diameter c
- Transmitter-receiver link at the origin
- Fixed and known useful signal level P_r
- Vehicles outside of the guard zone $[-r_0, r_0]$ generate interference
- Distance-based pathloss $g(r) = r^{-\eta}$, $r > r_0$
- Exponential fading over interfering links h_k and over the transmitter-receiver link h_t
- Instantaneous interference level $\mathcal{I} = \sum_k h_k g(x_k)$



Problem formulation

How does the deployment model for the vehicles (hardcore model vs a PPP of equal intensity λ) impact the performance of the transmitter-receiver link at the origin?

- Due to Campbell's theorem for stationary processes, the mean interference levels under the two models are equal

$$\mathbb{E}\{\mathcal{I}\} = 2\lambda \int_{r_0}^{\infty} x^{-\eta} \mathrm{d}x = \frac{2\lambda r_0^{1-\eta}}{\eta - 1}$$

- Interferers have correlated locations
  - Higher moments of interference and outage probability would be different under the two models
  - Cross-moments of interference would be different too and affect
    - Temporal performance, e.g., retransmission schemes
    - Spatial performance, e.g., multi-antenna receiver

Probability of outage

The calculation of the outage probability, $\text{Pr}_{\text{out}}(\theta) = \mathbb{P}\,(\text{SIR} \leq \theta)$, requires the probability generating functional (PGFL) of the hardcore process generating the interference

$$\text{Pr}_{\text{out}}(\theta) = 1 - \mathbb{E}_{\text{x}}\left\{\prod_k \frac{1}{1 + s\,x_k^{-\eta}}\right\},\ s = \frac{\theta}{P_r}$$

A lower bound can be obtained by the PPP of equal intensity [Stucki2014]

$$\text{Pr}_{\text{out}}(\theta) \geq 1 - e^{-2\lambda \int_{r_0}^{\infty}\left(1 - \frac{1}{1+sx^{-\eta}}\right)\mathrm{d}x}$$

An upper bound using the Jensen's inequality

$$\text{Pr}_{\text{out}}(\theta) \leq 1 - \exp\left(-\mathbb{E}_{\text{x}}\left\{\sum_k \log\left(1 + s x_k^{-\eta}\right)\right\}\right)$$

When the bounds become tight? – traffic conditions, system set-up, etc.

Approximate the probability of outage

Available methods

- Factorial moment expansion for the PGFL [Westcott1972]
- Horizontal shift of the outage probability due to PPP [Guo2015]
- Converting distance distribution to aggregate interference level distribution
- Calculate few moments of interference distribution and select suitable probability function to approximate it
    - variance & ratio of standard deviation over the mean
    - skewness
    - temporal & spatial Pearson correlation coefficient of interference

Moments of interference

The second moment of interference accepts contributions due to a single vehicle and also due to pairs

$$\mathbb{E}\{\mathcal{I}^2\} = \underbrace{\mathbb{E}\{h^2\}\int g^2(x)\,\lambda \mathrm{d}x}_{\frac{4\lambda r_0^{1-2\eta}}{2\eta-1}} + \mathbb{E}\{h\}^2 \int g(x)\,g(y)\,\underbrace{\rho^{(2)}(x,y)}\mathrm{d}x\mathrm{d}y$$

The calculation of the third moment of interference involves also triples of vehicles

$$\mathbb{E}\{\mathcal{I}^3\} = \mathbb{E}\{h^3\}\int g^3(x)\,\lambda \mathrm{d}x + 3\mathbb{E}\{h^2\}\,\mathbb{E}\{h\}\int g^2(x)\,g(y)\,\underbrace{\rho^{(2)}(x,y)}\mathrm{d}x\mathrm{d}y + \mathbb{E}\{h\}^3 \int g(x)\,g(y)\,g(z)\,\underbrace{\rho^{(3)}(x,y,z)}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Moment measures

Correlation properties for the hardcore model have been studied in the context of statistical mechanics [Salsburg1953]

- The pair correlation function (PCF) is

$$\rho^{(2)}(y,x)=\sum_{k=1}^{\infty}\rho_k^{(2)}(y,x)\,,y>x.$$

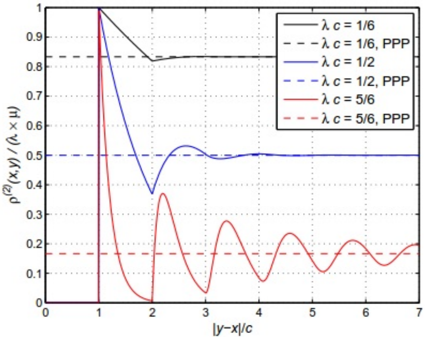
$$\rho_k^{(2)}(y,x)=\begin{cases}\lambda\sum_{j=1}^{k}\frac{\mu^j(y-x-jc)^{j-1}}{\Gamma(j)e^{\mu(y-x-jc)}}, & y\in(x+kc,x+(k+1)c)\,,k\geq 1\\ 0, & \text{otherwise.}\end{cases}$$

- Due to the 1D nature of the deployment, higher-order intensity measures are also available

$$\rho^{(3)}(x,y,z)=\frac{1}{\lambda}\rho^{(2)}(x,y)\,\rho^{(2)}(y,z)\,,\;x<y<z.$$

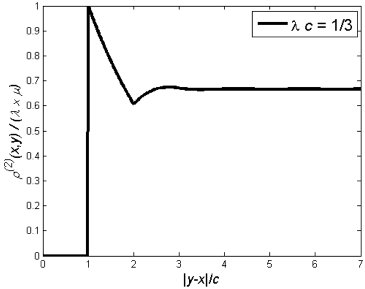
PCF

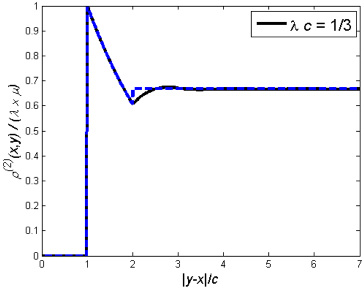
For small λc, the PCF of the hardcore process converges quickly (few multiples of the hardcore distance c) to the PCF λ^2 of a PPP of equal intensity



Simplifying the PCF

We will use the exact PCF only for small distances, e.g., up to $2c$, and the PCF due to a PPP of equal intensity beyond that distance − This approximation does not introduce much error for small λc





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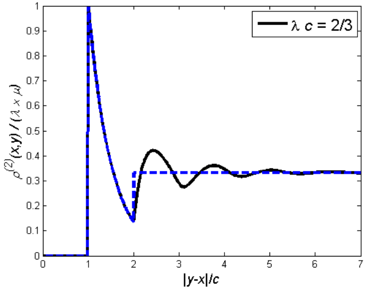
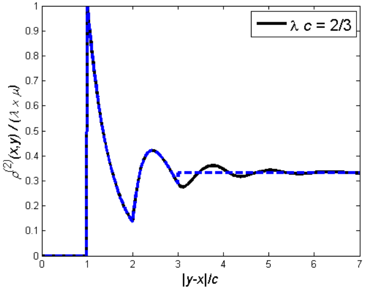




Approximation for the variance

Starting from

$$\mathbb{E}\{\mathcal{I}^2\} = \underbrace{\mathbb{E}\{h^2\}\int g^2(x)\,\lambda \mathrm{d}x}_{\frac{4\lambda r_0^{1-2\eta}}{2\eta-1}} + \mathbb{E}\{h\}^2 \int g(x)\,g(y)\,\underbrace{\rho^{(2)}(x,y)}\mathrm{d}x\mathrm{d}y$$

After substituting the approximation for the PCF, $\mu = \frac{\lambda}{1-\lambda c}$, and expanding around $\lambda c \to 0$ & $\frac{c}{r_0} \to 0$, the variance of interference becomes

$$\mathbb{V}\{\mathcal{I}\} \approx \underbrace{\frac{4\lambda r_0^{1-2\eta}}{2\eta - 1}}_{\mathrm{PPP}(\lambda)} \left(1 - \lambda c + \frac{1}{2}\lambda^2 c^2\right),$$

Remark 1

Since the mean interference levels under the two models are equal, the distribution of interference for small λc becomes more concentrated around the mean as compared to that due to a PPP of intensity λ.

Approximation for the skewness

Approximating the third-order intensity measure $\rho^{(3)}(x,y,z)$ similar to the PCF, and expanding the third-moment for $\lambda c \to 0$ & $\frac{c}{r_0} \to 0$ we get

$$\mathbb{S}\{\mathcal{I}\} \approx \underbrace{\frac{12\lambda r_0^{1-3\eta}}{3\eta - 1}\left(\frac{4\lambda r_0^{1-2\eta}}{2\eta - 1}\right)^{-\frac{3}{2}}}_{\text{PPP}(\lambda)}\left(1 - \frac{\lambda c}{2}\right)$$

Remark 2

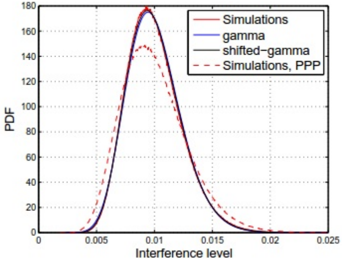
For small λc, the distribution of interference becomes more symmetric between the tails and remains positively-skewed as compared to that due to a PPP of intensity λ.

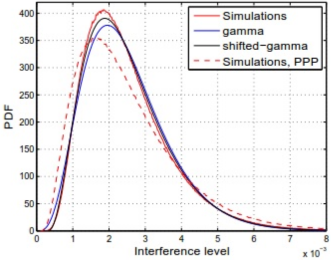
Remark 3

For fixed λc, the variance and the skewness of interference are proportional to $\frac{1}{\sqrt{\lambda r_0}}$. The error of PPP increases for smaller cell size and lower intensity of vehicles.

1. Due to the guard zone, the pathloss model is bounded, and the tails of interference strongly depend on the fading process [Pappas2015]
2. Skewness is positive for small λc

The Gamma and shifted-gamma probability distribution function (PDF) along with Rayleigh fading meet the above criteria
For fixed $\lambda c = 0.4$, a lower intensity λ is associated with higher skewness, and three moments clearly provide better fit than two.





Probability of outage

Both models have simple Laplace transforms

- gamma PDF

$$f_{\mathcal{I}}(x) \approx \frac{x^{k-1} e^{-x/\beta}}{\Gamma(k)\,\beta^k}, \quad \Pr_{\text{out}}(\theta) \approx 1 - (1 + s\beta)^{-k},$$

where $k = \frac{\mathbb{E}\{\mathcal{I}\}^2}{\mathbb{V}\{\mathcal{I}\}}$ and $\beta = \frac{1}{k}$.

- shifted-gamma PDF

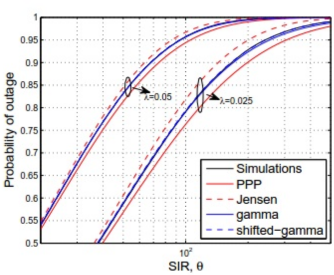
$$f_{\mathcal{I}}(x) \approx \frac{(x-\epsilon)^{k-1} e^{-(x-\epsilon)/\beta}}{\Gamma(k)\,\beta^k}, x \geq \epsilon, \ \Pr_{\text{out}}(\theta) \approx 1 - e^{-s\epsilon}(1 + s\beta)^{-k},$$

where $k = \frac{4}{\mathbb{S}\{\mathcal{I}\}^2}$, $\beta = \sqrt{\frac{\mathbb{V}\{\mathcal{I}\}}{k}}$ and $\epsilon = \mathbb{E}\{\mathcal{I}\} - k\beta$.

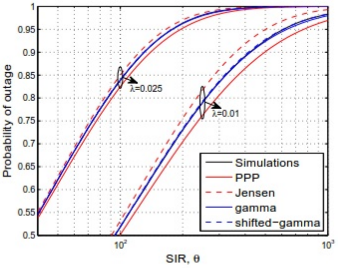
For fixed $\lambda c = 0.4$, the PPP starts to fail in the upper tail for microcells and also in macrocells with sparse flows

The two models (gamma, shifted-gamma) fit very well the simulations

Microcell $r_0 = 100$ m



Macrocell $r_0 = 250$ m



Mean delay

- We use static and independent realizations of interferers over the time slots to model low and high mobility respectively along microcells
- For independent realizations of interferers over time, the mean delay is the inverse of the probability of successful reception $\mathbb{E}\{D\} = \frac{1}{1-\text{Pr}_{\text{out}}(\theta)}$
- For static interferers over time

$$\mathbb{E}\{D\} \approx \sum_{t=0}^{\infty} \sum_{T=t}^{\infty} (-1)^t \binom{T}{t} (1+s\beta(t))^{-k(t)}$$
$$= \lim_{T_0 \to \infty} \sum_{t=0}^{T_0} (-1)^t \binom{T_0+1}{t+1} (1+s\beta(t))^{-k(t)} .$$

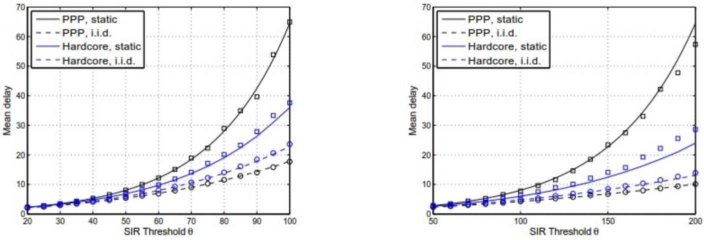
where $k(t) = \frac{\mathbb{E}\{\mathcal{I}(t)\}^2}{\mathbb{V}\{\mathcal{I}(t)\}}$, $\beta(t) = \frac{1}{k(t)}$.

$\mathbb{E}\{\mathcal{I}(t)\} = \frac{2\lambda r_0^{1-\eta} t}{\eta-1}, \mathbb{V}\{\mathcal{I}(t)\} \approx \frac{2\lambda r_0^{1-2\eta} t\left(1+t(1-\lambda c)^2\right)}{2\eta-1}.$

Numerical illustrations – Mean delay

For static interferers, the two models are characterized by different temporal correlation of interference

- The temporal correlation coefficient of interference for the PPP is $\frac{1}{2}$
- For small λc, the correlation coefficient for the hardcore model is approximately equal to $\frac{1}{2}(1 - \lambda c)$



(a) $\lambda = 0.05\,\text{m}^{-1}$ (b) $\lambda = 0.025\,\text{m}^{-1}$

Dual-branch maximum ratio combining (MRC)

Instantaneous interference at the two antennas

$$\mathcal{I}_1 = \sum\nolimits_i h_{1,i} g\left(x_i\right), \;\; \mathcal{I}_2 = \sum\nolimits_i h_{2,i} g\left(x_i\right)$$

Sum the post combining SIR conditioned on the interference vector

$$\mathbb{P}\{\mathrm{SIR} \geq \theta\} = \mathbb{E}_{\mathbf{I}}\left\{\mathbb{P}\left(\frac{h_{t,1}P_r}{\mathcal{I}_1} + \frac{h_{t,2}P_r}{\mathcal{I}_2} \geq \theta|\mathbf{I}\right)\right\}$$

Condition on the SIR for the second branch w [Tanbourgi2014]

$$\mathbb{P}\{\mathrm{SIR} \geq \theta\} = \mathbb{E}_{\mathbf{I},W}\left\{e^{-s_1\mathcal{I}_1}\right\} = \mathbb{E}_{\mathbf{I}}\left\{\int_0^\infty e^{-s_1(w)\mathcal{I}_1} f_{W|\mathcal{I}_2}(w)\,\mathrm{d}w\right\},$$

where $s_1(w) = \frac{\max\{0,\theta-w\}}{P_r}$

Dual-branch MRC

For Rayleigh fading, the conditional PDF is $f_{W|\mathcal{I}_2}(w) = \frac{\mathcal{I}_2}{P_r} e^{-s_2(w)\mathcal{I}_2}$, where $s_2(w) = \frac{w}{P_r}$

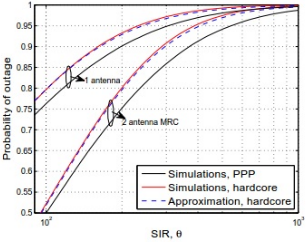
$$\begin{aligned}\mathbb{P}\{\mathsf{SIR} \geq \theta\} &= \frac{1}{P_r}\int_0^\infty \mathbb{E}_{\mathbf{I}}\left\{\mathcal{I}_2 e^{-s_1\mathcal{I}_1} e^{-s_2\mathcal{I}_2}\right\} \mathrm{d}w \\ &= \frac{1}{P_r}\int_0^\theta \mathbb{E}_{\mathbf{I}}\left\{\mathcal{I}_2 e^{-s_1\mathcal{I}_1} e^{-s_2\mathcal{I}_2}\right\} \mathrm{d}w + \frac{1}{P_r}\int_\theta^\infty \mathbb{E}_{\mathbf{I}}\left\{\mathcal{I}_2 e^{-s_2\mathcal{I}_2}\right\} \mathrm{d}w,\end{aligned}$$

Using the differentiation property of Laplace transform

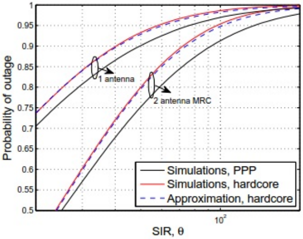
$$\mathbb{P}\{\mathsf{SIR} \geq \theta\} = P_r^k \left(P_r + \theta\beta\right)^{-k} + k\beta P_r^{2k} \int_0^\theta \frac{1 + \beta\left(\theta - w\right)\left(1 - \rho\right) \mathrm{d}w}{\left(P_r^2 + \theta\beta P_r + \left(\theta - w\right) w\beta^2 \left(1 - \rho\right)\right)^{k+1}}$$

Dual-branch MRC

- The outage probability prediction using the PPP model worsens with two antennas at receiver
- The outage probability due to a PPP is not anymore a bound (lower tail) – overall limited use



(a) $r_0 = 100\text{m}, \lambda = 0.025\text{m}^{-1}, c = 20\text{m}$



(b) $r_0 = 250\text{m}, \lambda = 0.01\text{m}^{-1}, c = 50\text{m}$

Conclusions

- The hardcore distance makes the interference distribution more concentrated around the mean and less skewed
- The PPP of equal intensity gives a lower bound for the outage probability with single-antenna receiver
  - The PPP bound fails when the coefficient of variation and the skewness of interference are high
  - Associated traffic scenarios are urban microcells & macrocells with sparse flow of vehicles
- The performance prediction of PPP worsens with temporal performance metrics and multi-antenna receivers because the hardcore distance impacts the correlation properties of interference too

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