Connectivity of random 1-dimensional networks

Vitaliy Kurlin, Durham
Mila Mihaylova, Lancaster
Simon Maskell, QinetiQ
http://maths.dur.ac.uk/~dma0vk
Initial motivations

- **dimension 1**: monitoring roads, boundaries of restricted areas
- **random**: automatic deployment along riversides difficult of access
Distributing along roads

Transmission radius $R >$ road width $W$. Then a 2-dim network of $(x_i, z_i)$ is connected iff the 1-dim network of $x_i$ with radius $\sqrt{R^2 - W^2}$ is connected.
Filling a 2-dimensional area

Distributing sensors along a snake-like path fills an area if the distance between adjacent branches $\Delta \leq R\sqrt{3}/2$. 
What is random?

- **common**: all (positions of) sensors have a prescribed density function
- **practical**: deploy sensors one by one along a trajectory of a vehicle, so the distance between successive sensors has a prescribed density
Our assumptions

- $R$ is a transmission radius
- sensors are deployed in $[0, L]$, a sink node is fixed at $x_0 = 0$
- $f_1, \ldots, f_n$ are independent densities of distances between sensors:
  \[ P(0 \leq x_i - x_{i-1} \leq R) = \int_0^R f_i(s)ds. \]
Connectivity and coverage

For a given probability and densities

- find a minimal number of randomly deployed sensors in \([0, L]\) such that the resulting network is connected;
- find a minimal number of random sensors such that the network is connected and covers \([0, L]\).
Key steps of our solution

- For arbitrary densities $f_1, \ldots, f_n$, compute the probability $P_n$ that the network of $n$ sensors is connected.
- Find estimates of $n$ such that $P_n$ is greater than the given probability.
Conditional probabilities

Given densities $f_1, \ldots, f_n$ of distances, $y_1, \ldots, y_n$ are naturally defined on $[0, L]$, but the network should be proper, i.e. all sensors are in $[0, L]$ or $\sum_{i=1}^{n} y_i \leq L$.

We compute the probability that the network is connected if it is proper.
2-sensor networks

A network of 2 sensors with distances $y_1 = x_1 - 0$, $y_2 = x_2 - x_1$ is represented by $(y_1, y_2) \in \{y_1, y_2 \geq 0 \ y_1 + y_2 \leq L\}$. 

\[ L \]

\[
\begin{array}{c}
\text{connected networks} \\
(y_1, y_2)
\end{array}
\]

\[
\begin{array}{c}
\text{proper networks}
\end{array}
\]

\[ R \]

\[ L \]
The probability of connectivity is

\[ P_{2}^{u} = \begin{cases} 
2 \left( \frac{R}{L} \right)^{2} & \text{if } R \leq \frac{L}{2}, \\
4 \left( \frac{R}{L} \right) - 2 \left( \frac{R}{L} \right)^{2} - 1 & \text{if } R \geq \frac{L}{2}.
\end{cases} \]
Connectivity Theorem

The probability of connectivity is

\[ P_n = \nu_n(R, L)/\nu_n(L, L), \] where

\[ \begin{align*}
\nu_0(r, l) &= 1, \quad r, l > 0; \\
\nu_n(r, l) &= 0, \quad r \leq 0 \text{ or } l \leq 0; \\
\nu_n(r, l) &= 1, \quad r \geq l > 0, \quad n > 0; \\
\nu_n(r, l) &= \int_0^r f_n(s)\nu_{n-1}(r, l - s)ds, \quad r < l.
\end{align*} \]
\[ P_n = \frac{\nu_n(R, L)}{\nu_n(L, L)} \]

+ closed formula for finite networks
+ arbitrary different densities

− can be computationally difficult
+ explicit for important distributions
+ implies simple estimates for \( n \)
The recursive function

\( v_n(r, l) \) is the probability that random distances having densities \( f_1, \ldots, f_n \) satisfy \( \sum_{i=1}^{n} y_i \leq l \) and \( 0 \leq y_i \leq r \), e.g.

\[
\begin{align*}
v_1(r, l) &= \int_0^r f_1(s) \, ds, \quad r < l, \\
v_2(r, l) &= \int_0^r f_2(s) v_1(r, l - s) \, ds.
\end{align*}
\]

\( v_n(L, L) \): the network is proper,
\( v_n(R, L) \): the network is connected.
Coverage Theorem

The probability of coverage is

\[
\frac{(v_n(R, L) - v_n(R, L - R))}{v_n(L, L)}.
\]

\[
\frac{v_n(R, L)}{v_n(L, L)}: \text{connected if proper on } [0, L],
\]

\[
v_n(R, L - R)/v_n(L, L): \text{connected network if proper on } [0, L - R].
\]
Uniform Corollary

If all $f_i = 1/L$ then the probability is

$$P_n^u = \sum_{i=0}^{i<L/R} (-1)^i \binom{n}{i} (1 - iR/L)^n.$$ 

$P_1^u = R/L$: connected with the sink.

$$P_2^u = \begin{cases} 2(R/L)^2 & \text{if } R \leq L/2, \\ 4(R/L) - 2(R/L)^2 - 1 & \text{if } R \geq L/2. \end{cases}$$
Uniform case: simulations

$L = 1\text{km}, \ R = 50\text{m}, \ n \leq 200\text{ sensors.}$
Uniform case: estimate

Set $Q = (L/R) - 1$. The network is connected with a probability $p > 2/3$ if

$$n \geq \frac{3}{2}(1 - Q) + \sqrt{\frac{(3Q - 1)^2}{4}} + 6Q^2 \left(\frac{Q}{1 - p} - 1\right).$$

<table>
<thead>
<tr>
<th>Transmission Radius, m.</th>
<th>200</th>
<th>100</th>
<th>50</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min Number of Sensors</td>
<td>29</td>
<td>69</td>
<td>157</td>
<td>349</td>
</tr>
<tr>
<td>Estimate of Min Number</td>
<td>83</td>
<td>283</td>
<td>905</td>
<td>2610</td>
</tr>
</tbody>
</table>
Uniform case: conclusions

- less effective than non-random
- rough estimate, not optimal
+ quadratic estimate is used later
+ can be improved using Taylor approximations of degrees 4, 5
+ non-trivial inequalities $0 \leq P_n^u \leq 1$
A constant density: graph

Let \( f = \frac{1}{(b - a)} \) over \([a, b] \subset [0, L]\).

\( n = 1: \ P(0 \leq y_1 \leq R) = \frac{(R - a)}{(b - a)}. \)
Constant Corollary

If all $f_i = 1 / (b - a)$ then the probability is

$$P^c_n = \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} (L - a(n - k) - Rk)^n}{\sum_{k=0}^{n} (-1)^k \binom{n}{k} (L - a(n - k) - bk)^n}.$$

$$P^c_1 = \frac{(L - a) - (L - R)}{(L - a) - (L - b)} = \frac{R - a}{b - a}.$$
Constant case: simulations

$L = 1\text{km}, \ R = 50\text{m}, \ a = 10\text{m}, \ b = 80\text{m}.$
Constant case: estimate

The network is connected with a probability $p$ if $\frac{a + b}{2} \leq R \leq b$ and

\[ n \geq \max \left\{ \frac{3}{2} + \sqrt{\frac{1 + 5p}{1 - p}}, 1 + \frac{L - b}{a} \right\} . \]

For all $p$ not too close to 1, the 2nd estimate holds: $L + a - b \leq an < L$. 
Constant case: conclusions

Constant density over \([0.2R, 1.6R]\).

<table>
<thead>
<tr>
<th>Transmission Radius, m.</th>
<th>200</th>
<th>150</th>
<th>100</th>
<th>50</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min Number of Sensors</td>
<td>14</td>
<td>19</td>
<td>30</td>
<td>63</td>
<td>132</td>
</tr>
<tr>
<td>Estimate of Min Number</td>
<td>18</td>
<td>27</td>
<td>43</td>
<td>93</td>
<td>193</td>
</tr>
<tr>
<td>Max Number of Sensors</td>
<td>25</td>
<td>34</td>
<td>50</td>
<td>100</td>
<td>200</td>
</tr>
</tbody>
</table>

- minimal practical assumptions
- very simple effective estimate
- non-trivial inequalities \(0 \leq P_n^c \leq 1\)
Exponential Corollary

If the distances between successive sensors have the density $f(s) = ce^{-\lambda s}$ on $[0, L]$, then the probability of connectivity is $P^e_n = \frac{v_n(R, L)}{v_n(L, L)}$, $v_n(r, l) = \sum_{i=0}^{i<l/r} (-1)^i \binom{n}{i} e^{-i\lambda r} \frac{e^{-\lambda(l-ir)}}{\lambda^n} \left(1 - e^{-\lambda(l-ir)} \sum_{j=0}^{n-1} \frac{\lambda^j(l-ir)^j}{j!}\right)$.
Exponential conclusions

Estimate: as in the uniform case.

The denominator tends to 0 fast:

\[ v_n(L, L) = 1 - e^{-\lambda L} \sum_{j=0}^{n-1} \frac{(\lambda L)^j}{j!} \]

- unpractical: throw on the alert
- sensors are too close to each other
Normal distribution

If \( f(s) = \frac{c}{\sigma \sqrt{2\pi}} e^{-(s-\mu)^2/2\sigma^2} \) on \([0, L]\)
then the distances between successive sensors are close to the mean \( \mu \), e.g.
very likely to be in \([\mu - 3\sigma, \mu + 3\sigma]\)

Reasonable to assume: \( \mu < R, n\mu < L \).
Normal case: estimate

The network with normal distances is connected with a given probability \( p \) if

\[
n \leq \min \left\{ \frac{p(1-p)}{\varepsilon}, \frac{\left( \sqrt{4\mu L + \sigma^2\Phi^{-2}(p)} - \sigma\Phi^{-1}(p) \right)^2}{4\mu^2} \right\}
\]

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds, \quad \varepsilon = \Phi \left( -\frac{\mu}{\sigma} \right) + 1 - \Phi \left( \frac{R - \mu}{\sigma} \right).
\]
Normal case: example

\[ \mu = 0.6R, \sigma = 0.1R, p = 0.9975. \]

Then \( \Phi^{-1}(p) \approx 2.8, \varepsilon \approx 0.000063. \)

\( R = 25\text{m}: \ n \leq p(1 - p)/\varepsilon \approx 40. \)

\( R \geq 50\text{m}: \text{the 2nd estimate is close to} \]

\[ \frac{L}{\mu} \approx \frac{\left( \sqrt{4\mu L + \sigma^2\Phi^{-2}(p)} - \sigma\Phi^{-1}(p) \right)^2}{4\mu^2}. \]
Let $L = 1\text{km}$, $\mu = 0.6R, \sigma = 0.1R$.

<table>
<thead>
<tr>
<th>Transmission Radius, m.</th>
<th>200</th>
<th>150</th>
<th>100</th>
<th>50</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate of Max Number</td>
<td>7</td>
<td>11</td>
<td>16</td>
<td>33</td>
<td>40</td>
</tr>
</tbody>
</table>

6 non-random sensors are enough for the radius $R = 150\text{m}$: $6/11 \approx \mu/R$. 
All cases: conclusions

- exponential: too dense networks
- normal: ideal density $\Rightarrow$ ideal results
- uniform: a useful theoretical exercise
+ constant over $[a, b]$: very reasonable
+ more complicated: piecewise constant?
Ideas of proofs

- induction on the number of sensors: adding 1 sensor keeps connectivity if it is close to the previous one
- the key probability $v_n(r, l)$ is an iterated convolution of densities computed by the Laplace transform
More explicit formulae

- heterogeneous networks: distances have different constant densities
- building densities from blocks: any piecewise constant density
- more can be produced easily
A 3-step density: graph

\[ C, R \text{ are chosen so that } \int_0^L f(s) \, ds = 1. \]
A 3-step density: formula

The probability of connectivity is $P_n = \sum_{m=0}^{n} \sum_{k_1=0}^{m} \sum_{k_2=0}^{n-m} \frac{(-1)^{k_1+k_2}(L - (2k_1 + k_2 + n - m)R/2)^n}{d_m k_1!(m - k_1)!k_2!(n - m - k_2)!}

\sum_{m=0}^{n} \sum_{k_1=0}^{m} \sum_{k_2=0}^{n-m} \frac{(-1)^{k_1+k_2}(L - (2k_1 + 2k_2 + n - m)R/2)^n}{d_m k_1!(m - k_1)!k_2!(n - m - k_2)!}

$d_m = C^{-m}(1/R - C)^{m-n}$, the sums are over all $m, k_1, k_2$ if the terms $> 0$. 
A 3-step density: simulations

Let $L = 1\text{km}$, $R = 50\text{m}$, $C = 0.9/R$. 

![Graph showing the 3-step density simulations]
A 3-step density: table

<table>
<thead>
<tr>
<th>Transmission Radius, m.</th>
<th>250</th>
<th>200</th>
<th>150</th>
<th>100</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min Number of Sensors</td>
<td>12</td>
<td>17</td>
<td>25</td>
<td>44</td>
<td>105</td>
</tr>
</tbody>
</table>

+ flexible practical assumptions
+ reasonable estimates of min number
+ non-trivial inequalities $0 \leq P_n \leq 1$
Open problem 1

Compute the exact probability of connectivity if the distances between successive sensors have a truncated normal density on $[0, L]$. 
Open problem 2

For a given segment $[0, L]$ and number $n$ of sensors, find an optimal density of distances between successive sensors to maximise the probabilities of connectivity and coverage.
Open problem 3

Compute the probabilities of connectivity and coverage if sensors are randomly deployed along a non-straight trajectory filling a 2-dimensional area.