Networks embedded in Lorentzian spaces

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September 11, 2015

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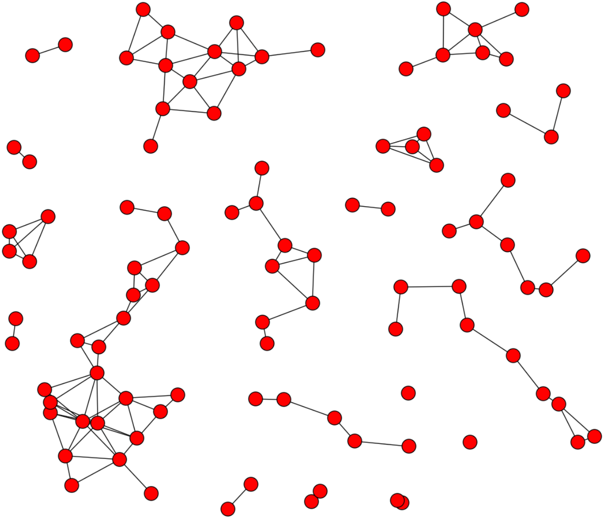
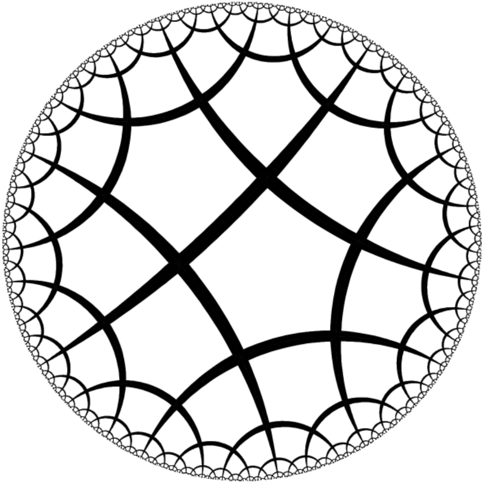
What do we mean by network geometry??

- Nodes have coordinates in some space
- Whether an edge exists is a function of the node's coordinates

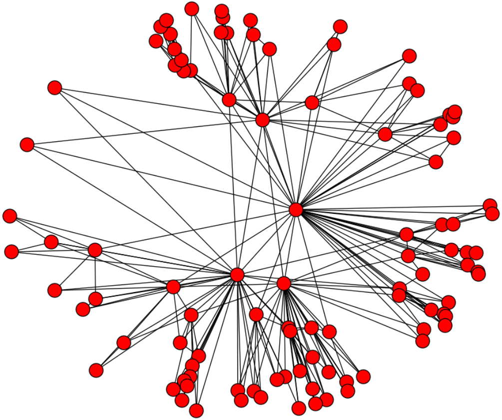
2D Random Geometric Graph



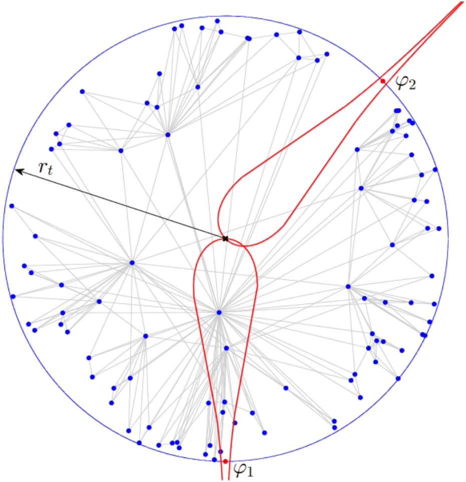
Hyperbolic geometry



Hyperbolic network geometry



Hyperbolic network geometry



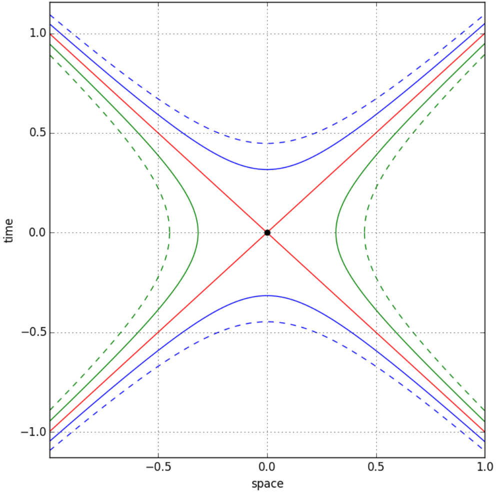
Riemannian and Lorentzian Geometry

- Euclidean space, hyperbolic space, the surface of a sphere etc. are all Riemannian geometries
- The *eigenvalues* of the metric are all positive
- Euclidean space: $ds^2 = dx^2 + dy^2 + dz^2$
  Sphere surface: $ds^2 = d\theta^2 + \sin(\theta) d\phi^2$

Riemannian and Lorentzian Geometry

- In this talk - Lorentzian Geometry
- One eigenvalue of the metric is negative
- The most well known example is Minkowski space - the geometry of special relativity where $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
- Distances can be positive, negative or 0.

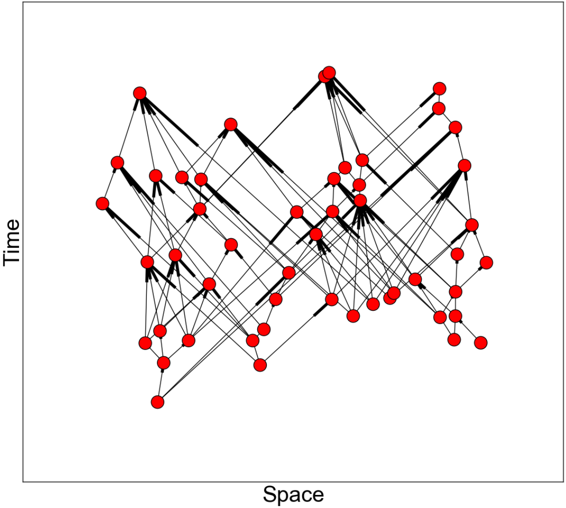
Simplest example: Minkowski space



Causal Sets

- A link back to physics - causal sets
- Approach to quantum gravity where spacetime is a set of discrete points that approximate the continuous spacetime we perceive
- Relations in the causal set correspond to timelike separation

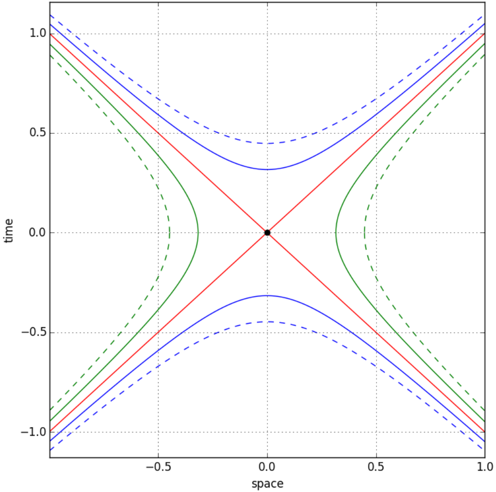
1+1 dimensional causal set



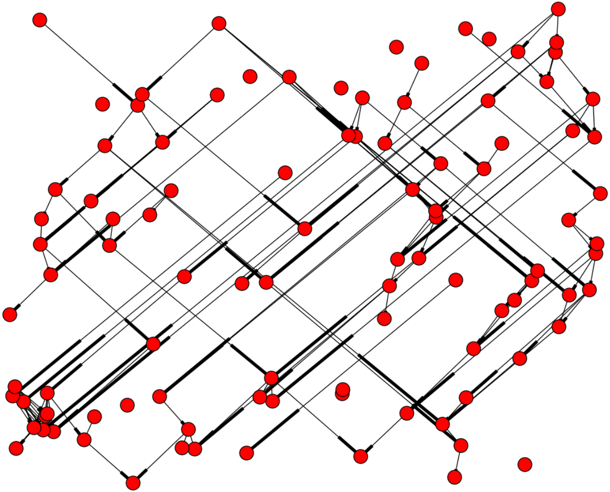
Causal Sets

- Causal sets $\rightarrow$ Directed Acyclic Graph.
- Interesting properties - Lorentz invariance.
- Geodesic is the *longest chain* and not the shortest.
- Unbounded number of nearest neighbours.

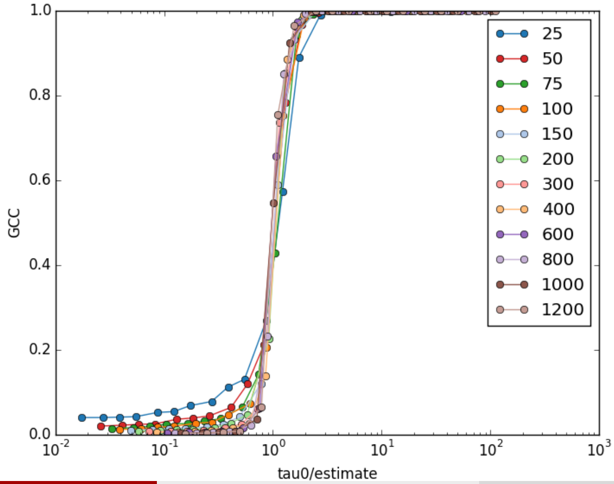
Minkowski space



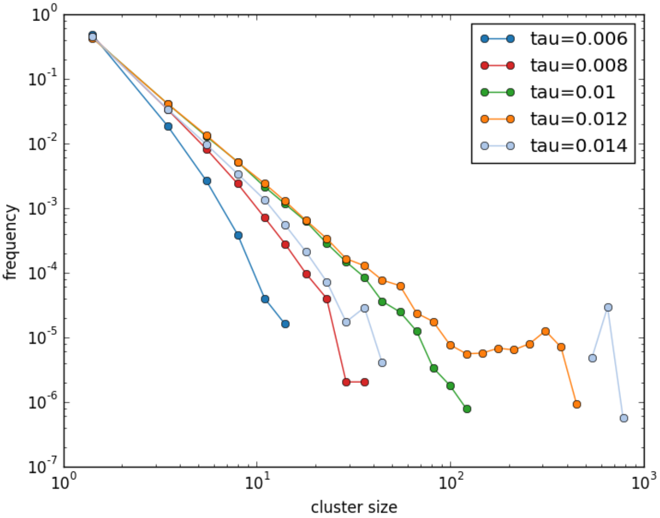
Lorentzian Random Geometric Graphs



Lorentzian Random Geometric Graphs - phase transition



Lorentzian Random Geometric Graphs - scaling



Real networks

- Lots of real networks have some kind of approximate Euclidean/hyperbolic geometry
- Do any have approximate Lorentzian geometry?
- Clues - nodes exist at a certain point in time, and edges represent causal connections between them
- Network forms a Directed Acyclic Graph (DAG)

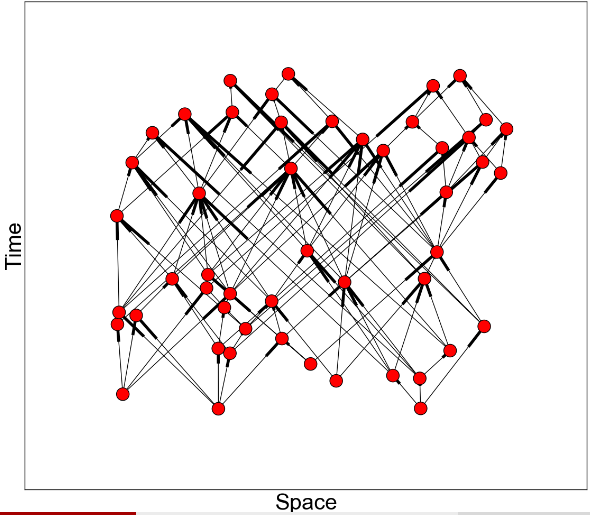
Let's speculate: Citation Networks?

- Documents appear at a particular time.
- They (usually) can only cite something written in the past, and so form a DAG.
- Academic papers - spatial dimensions could correspond to different academic fields?
- Citations in the same field have a shorter timespan than citations to different fields - suggestive of this kind of geometry?

Citation Networks

- Assume there is some Lorentzian geometry and see what we can measure
- I said before that the *longest chain* is a good approximation to the geodesic and not the shortest
- Traditional centrality measures use the shortest path as a measure of distance - what if we use the longest?

Example - 1+1 random Minkowski space network



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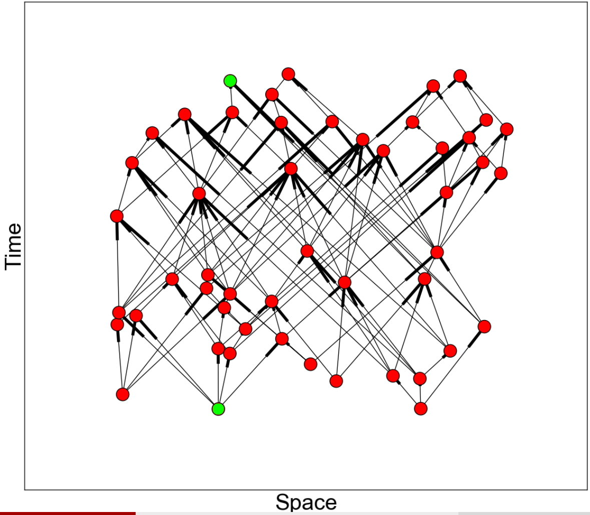

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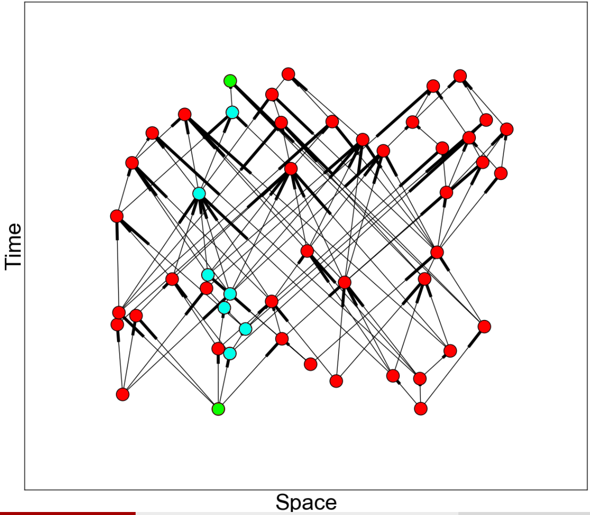

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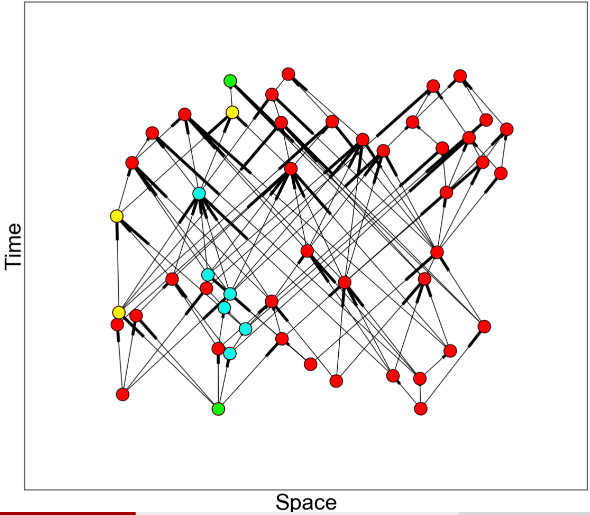

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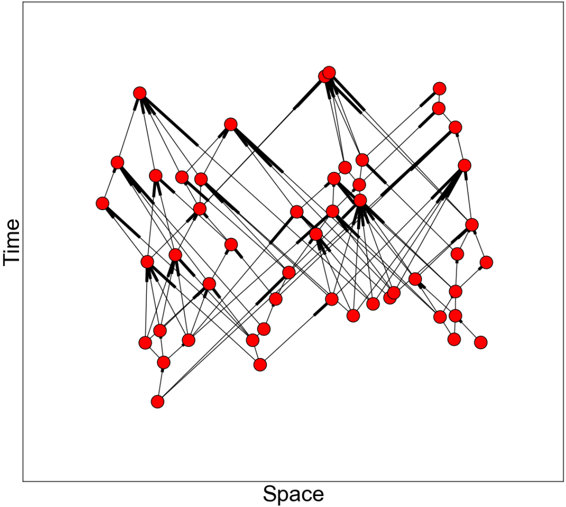

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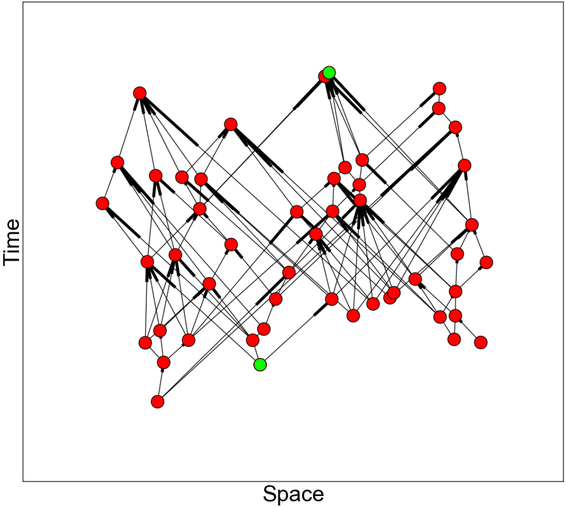

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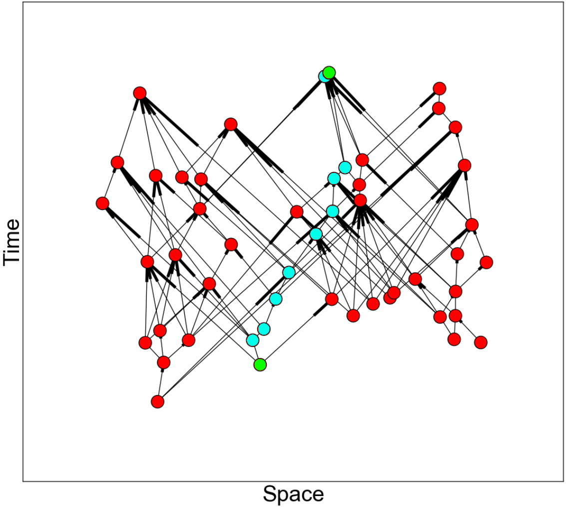

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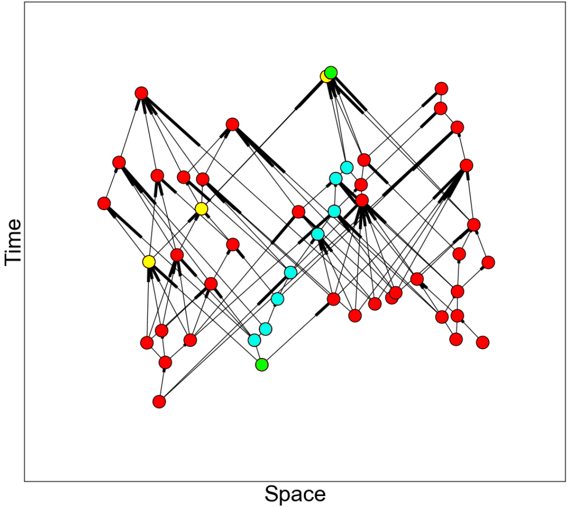


Test on small citation network

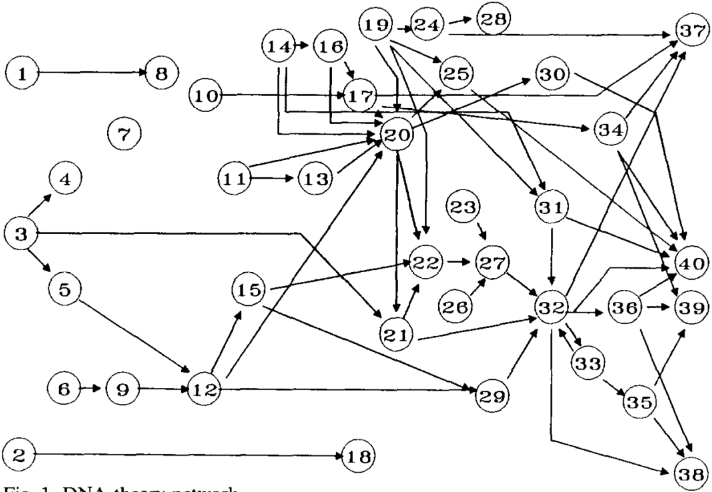
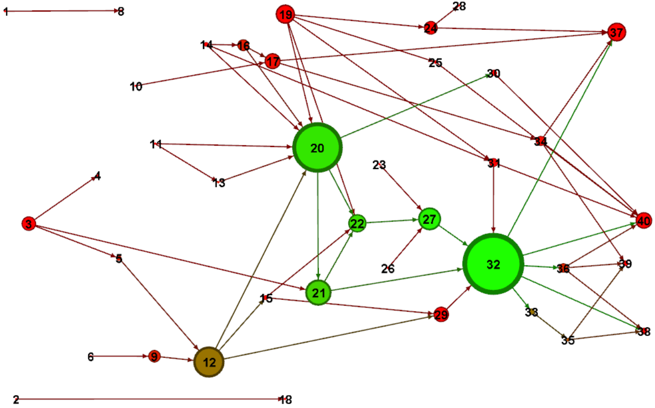


Fig. 1. DNA theory network.

Test on small citation network



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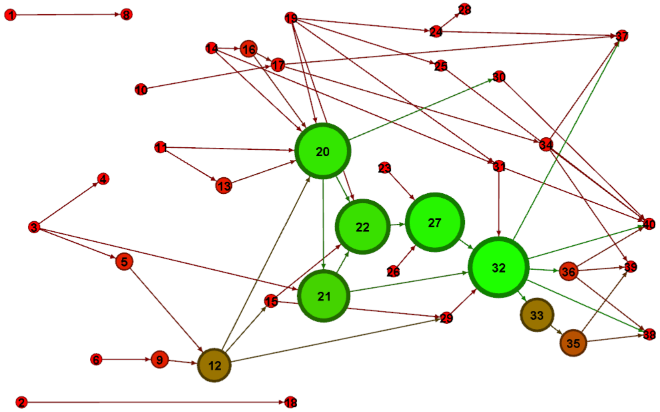
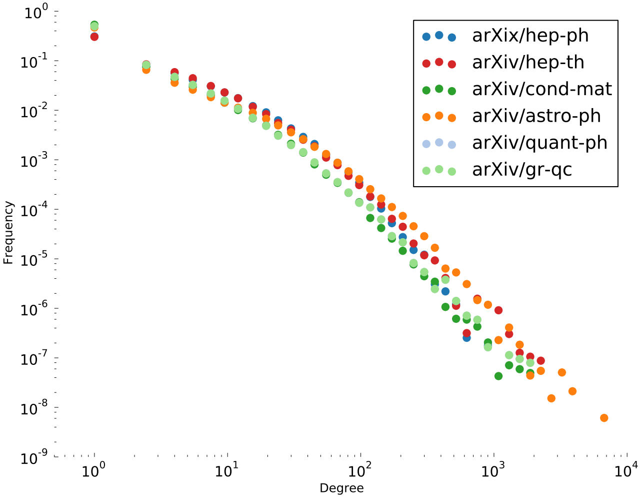


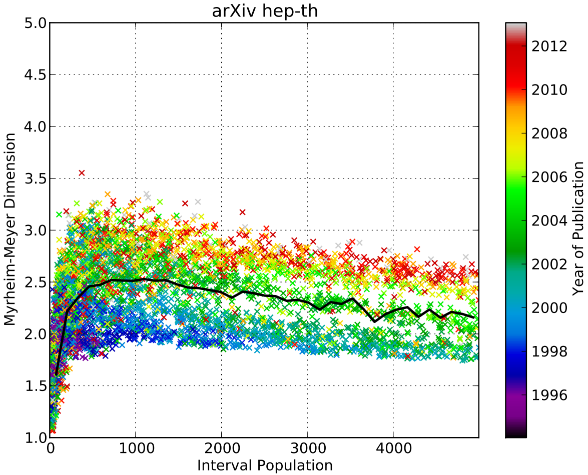
Characterising the Lorentzian Space

- Can we characterise the space the network is embedded in?
- Simplest example - Minkowski space - only 1 parameter - dimension.
- By measuring how the population of the network scales with a length scale (longest path) we can estimate a dimension.
- Does this tell us anything interesting about real networks?

Degree distributions similar



Dimension estimates different



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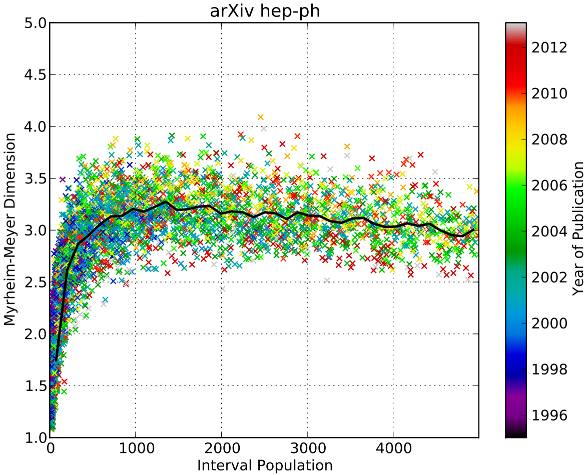

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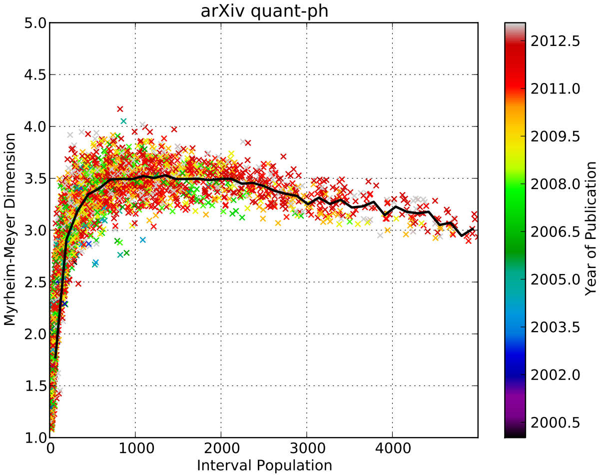

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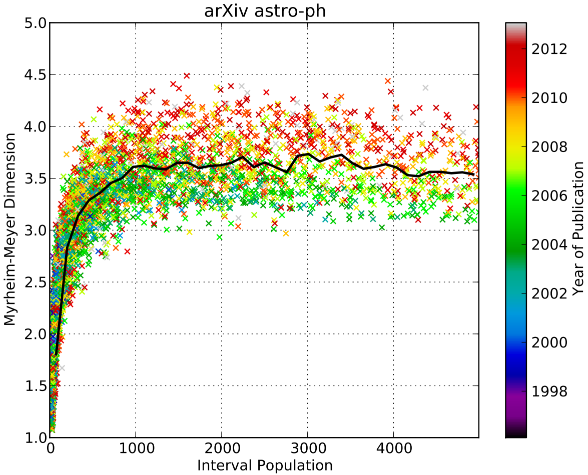
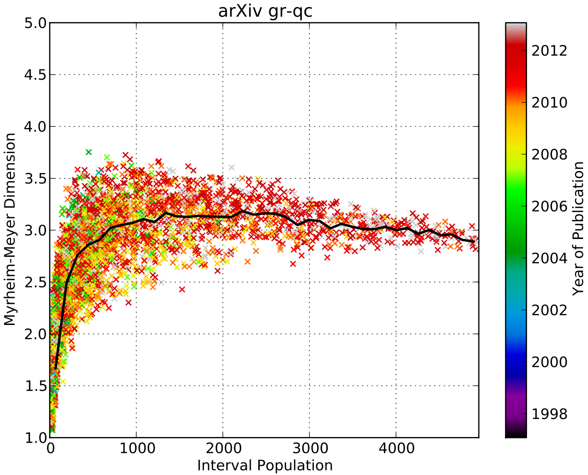


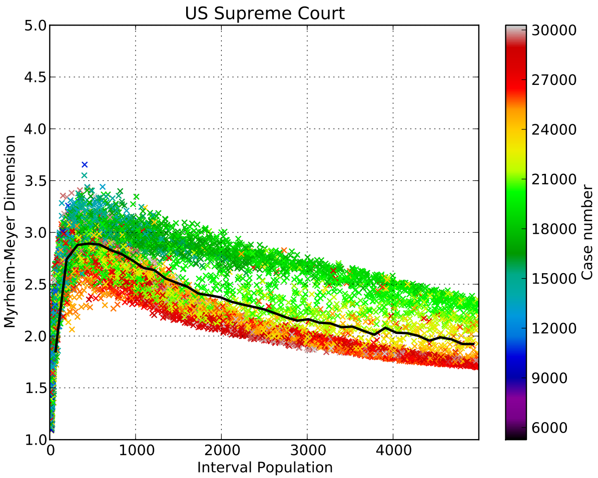
Dimension estimates different



Discussion

- Networks with very similar degree distributions and clustering have very different ‘dimensions’.
- This method provides a new way of quantifying a characteristic of the structure of citation networks.
- Measuring properties of the embedding space (dimension, curvature etc.) is needed before we can try to embed the network (i.e. ‘find’ the coordinates of each node)

US Supreme Court Citation Network



Discussion

- Citation behaviour changes over time.
- Citation networks from different fields have very different structure - can we measure different kinds of citation behaviour?
- Possible interpretation - high dimension $\rightarrow$ more diverse citation behaviour.

Future direction?

- How can we better determine when it's useful to say a network has some geometric properties?
- What other ways of characterising those spaces are there? And how do we interpret them?
- What can we learn about individual nodes in a network by finding a way to embed them in a space?

Bibliography I

D. Krioukov et al.
Hyperbolic Geometry of Complex Networks,
10.1103/PhysRevE.82.036106, 2010

F. Dowker,
Introduction to Causal Sets and their phenomenology.
General Relativity and Gravitation, 2013

N. Hummon, P. Doreian
Connectivity in a citation network: The development of DNA theory
Social Networks, 11:39–63, 1989.

J.R. Clough, T.S. Evans
What is the Dimension of Citation Space?,
http://arxiv.org/abs/1408.1274