Border effects in ad-hoc networks

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Outline

1. Random geometric graph model of wireless networks

2. Connectivity in convex domains
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3. Non-convex domains

Random geometric graphs

Introduced in 1961 by E. N. Gilbert:

> *Recently random graphs have been studied as models of communications networks. Points (vertices) of a graph represent stations; lines of a graph represent two-way channels. . . . To construct a random plane network, first pick points from the infinite plane by a Poisson process with density D points per unit area. Next join each pair of points by a line if the pair is separated by distance less than R.*

Then:

Communications networks Many authors, since 1980s

Connectivity threshold Penrose (1997), Gupta & Kumar (1999)

Books:

- **Meester & Roy (1996)** Continuum percolation
- **Penrose (2003)** Random geometric graphs
- **Franceschetti & Meester (2008)** Random networks for communication
- **Walters (2011)** Random geometric graphs (survey article)
- **Haenggi (2012)** Stochastic geometry for wireless networks

Network considerations

Mesh architectures Multihop connections rather than direct to a base station: Reduces power requirements, interference, single points of failure.

Random node locations In many applications (sensor, vehicular, swarm robotics, disaster recovery, ...) device locations are unplanned and/or mobile.

Network characteristics Full connectivity, k-connectivity (resilience; OG, CPD and JPC, EPL 2013), betweenness centrality (importance, overload; A.P. Giles, OG and CPD, ICC 2015), algebraic connectivity (synchronisation).

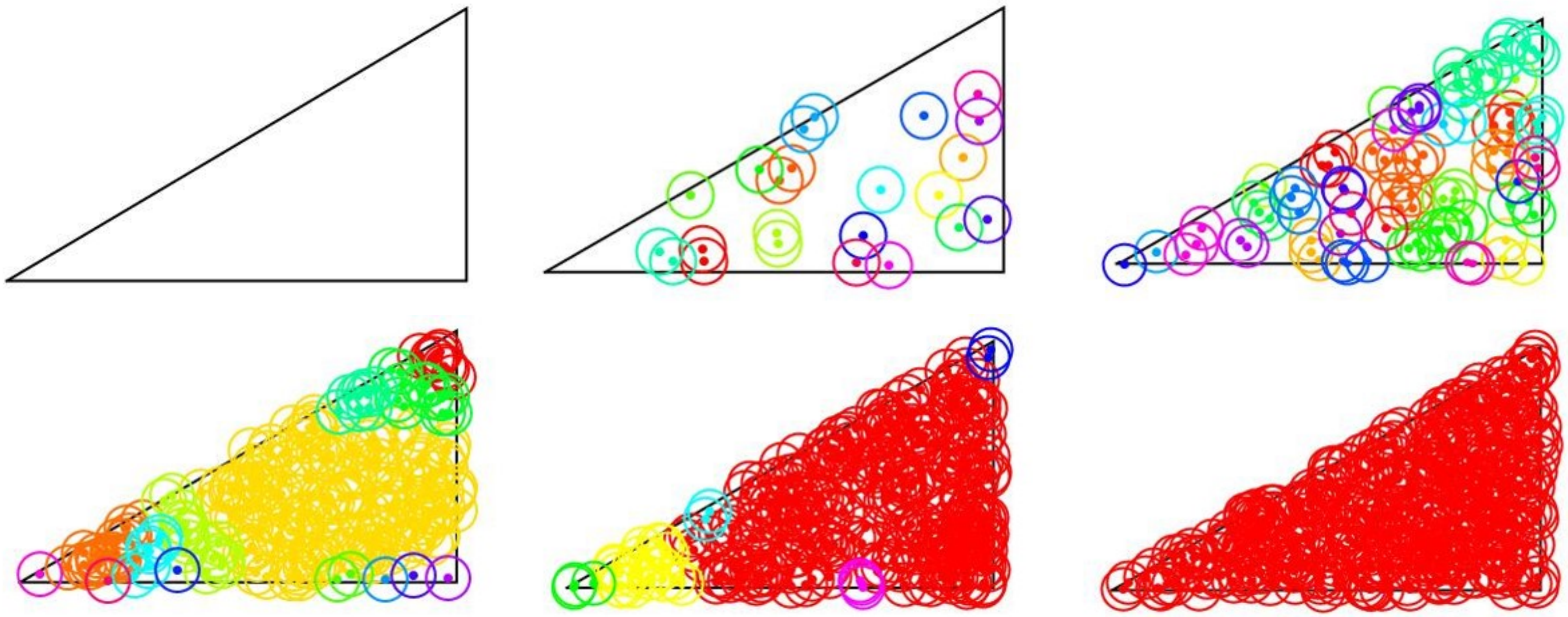
Useful extensions:

Random connection models Extra randomness: Form a link with (iid) probability $H(r) \in [0, 1]$, a function of the mutual distance r.

Line of sight condition Impenetrable and/or reflecting boundaries: Particular relevance to networks using millimetre waves.

Etc : Interference (later) anisotropic connections (OG, CPD and JPC, TWC 2014), heterogeneous networks, mobility, dynamic networks, trust...

Example: A triangle



Isolated nodes occur mostly near the corners...

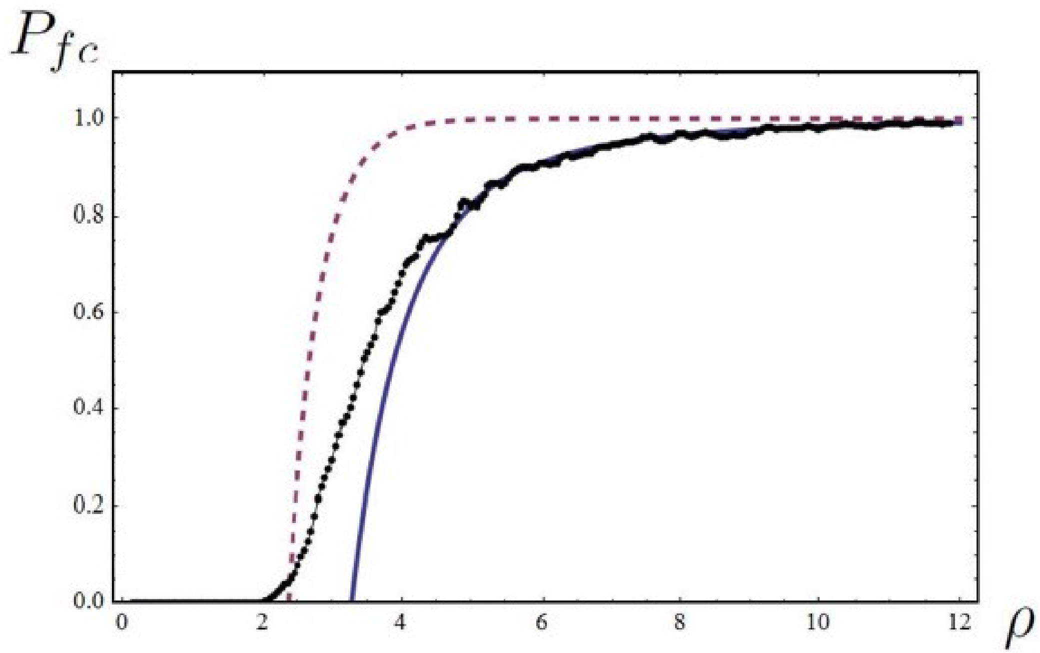
Dependence on density and geometry

We see two main transitions as density increases:

Percolation Formation of a cluster comparable to system size:
Largely independent of geometry.

Connectivity All nodes connected in multi-hop fashion:
Strongly dependent on geometry.

What is the full connection probability as a function of density and geometry?



Previous results

Mathematically rigorous results are in the limit of many nodes, taking appropriate scaling for r_0, L and/or ρ.

For the random geometric graph in dimension $d \geq 2$, it was shown by Penrose, and by Gupta & Kumar, that the r_0 threshold for **connectivity** is almost always the same as for **isolated nodes**.

In turn, isolated nodes are local events, so described by a limiting Poisson process: The probability of a node having degree k is given by

$$P(k) = \frac{\mathcal{K}^k}{k!} e^{-\mathcal{K}}$$

where $\mathcal{K}$ is the mean degree, equal to $\rho\pi r_0^2$ for the 2D RGG. This leads to

$$P_{fc} \approx \exp\left[-\rho V e^{-\rho\pi r_0^2}\right]$$

where V is the "volume" (ie area) of the domain.

At fixed probability, V needs to increase exponentially with ρ

Random connection models

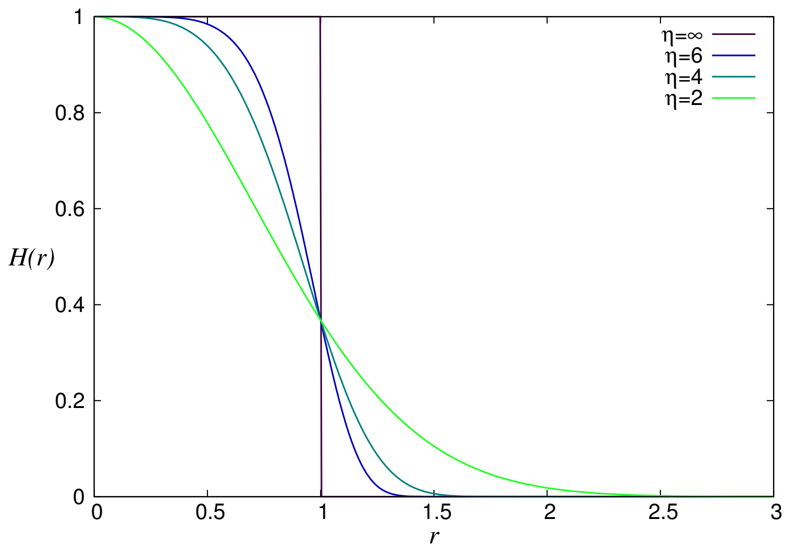
The connection function is the complement of the outage probability,

$$H(r) = \mathbb{P}(\log_2(1 + SNR\ |h|^2) > R_0)$$

neglecting interference, with $SNR \propto r^{-\eta}$, path loss exponent $\eta \in [2, 6]$, rate R_0. Simplest is Rayleigh fading (diffuse signal), for which the channel gain $|h|^2$ is exponentially distributed, giving

$$H(r) = \exp[-(r/r_0)^\eta]$$

Similar, though more involved: MIMO, Rician (specular plus diffuse), ...



Connectivity in the random connection model

The formula generalises naturally

$$P_{fc} \approx \exp\left[-\int \rho e^{-\rho \int H(r_{12}) d\mathbf{r}_1} d\mathbf{r}_2\right]$$

where ρ is the density, and the integrals are over the domain $\mathcal{V} \subset \mathbb{R}^d$. It has been proved under specific conditions (see Penrose, 2015):

- In the limit of infinitely many nodes
- Isolated nodes are approximately Poisson for a large class of connection functions (not annulus).
- Connectivity is equivalent to absence of isolated nodes for a smaller class (in particular, compact support).
- The domain is a d-dimensional cube.

We assume the above expression provides a useful approximation P_{fc} when these assumptions are relaxed, ie finite density, exponentially decaying connection functions, general convex domains in two or three dimensions.

Open problem: 1D networks with the random connection model.

Convex polyhedra, etc

For large ρ, we expect the domination by the regions of small connectivity mass

$$M(\mathbf{r}_2) = \int H(r_{12}) d\mathbf{r}_1$$

Exactly on the boundary, this is given by

$$M_B = H_{d-1}\omega_B$$

where

$$H_m = \int_0^\infty H(r) r^m dr$$

is the mth moment, and ω_B is the (solid) angle associated with the boundary component B, eg $\pi/2$ for a right angled corner, π for an edge. We analyse the vicinity of boundaries more carefully to obtain...

General formula

$$P_{fc} \approx \exp\left[-\sum_B \rho^{1-i_B} G_B V_B e^{-\rho\omega_B H_{d-1}}\right]$$

where i_B is the boundary codimension, V_B is its $d-i$ dimensional volume, and G_B is the geometrical factor

G_B	$i=0$	$i=1$	$i=2$	$i=3$
$d=2$	1	$\frac{1}{2H_0}$	$\frac{1}{H_0^2 \sin\omega}$	
$d=3$	1	$\frac{1}{2\pi H_1}$	$\frac{1}{\pi^2 H_1^2 \sin(\omega/2)}$	$\frac{4}{\pi^2 H_1^3 \omega \sin\omega}$

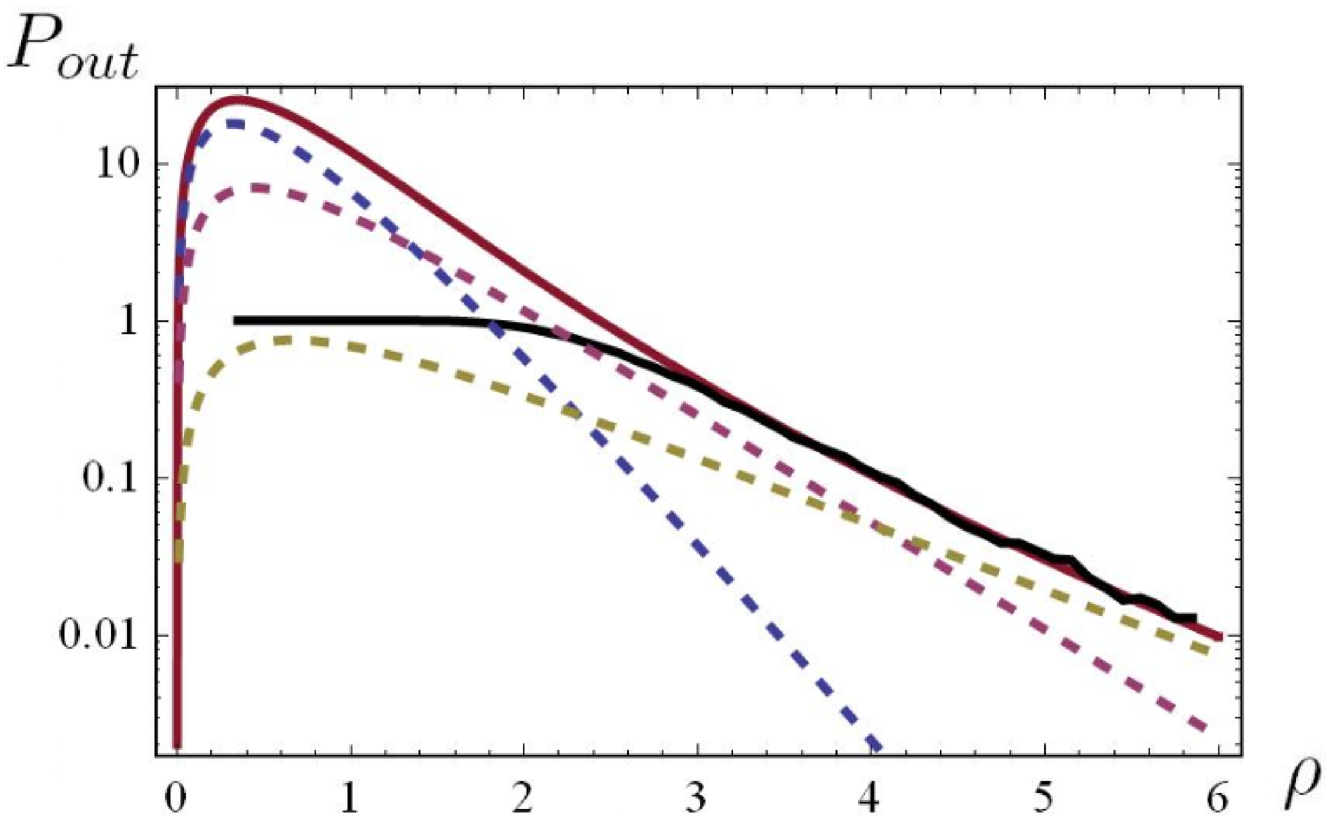
where the 3D corner has a right angle.

Curved boundaries? To leading order, curvature can be neglected.

Example: A square

The previous formula gives

$$1 - P_{fc} \approx L^2 \rho e^{-\pi\rho} + \frac{4L}{\sqrt{\pi}} e^{-\frac{\pi\rho}{2}} + \frac{16}{\pi\rho} e^{-\frac{\pi\rho}{4}}$$



Higher order terms

Expanding around the boundary points yields higher terms as well. In 2D:

Bulk

$$\rho A e^{-2\pi\rho H_1}$$

Edge

$$L e^{-\rho\pi H_1}\left[\frac{1}{2H_0} - \frac{\tilde{H}_{-2}}{8\rho^2 H_0^4} + \dots\right]$$

Corner, angle ω

$$e^{-\rho\omega H_1}\left[\frac{1}{\rho H_0^2 \sin\omega} - \frac{H(0)(2\cos\omega + 1)}{\rho^2 H_0^4 \sin^2\omega} - \frac{2\tilde{H}_{-2}}{\rho^3 H_0^5 \sin\omega} + \dots\right]$$

where $\tilde{H}_{-2}$ is a regularised negative second moment, equal to

$$\int_0^\infty \frac{H'(r)}{r} dr + \sum_k \frac{H(r_k^+) - H(r_k^-)}{r_k}$$

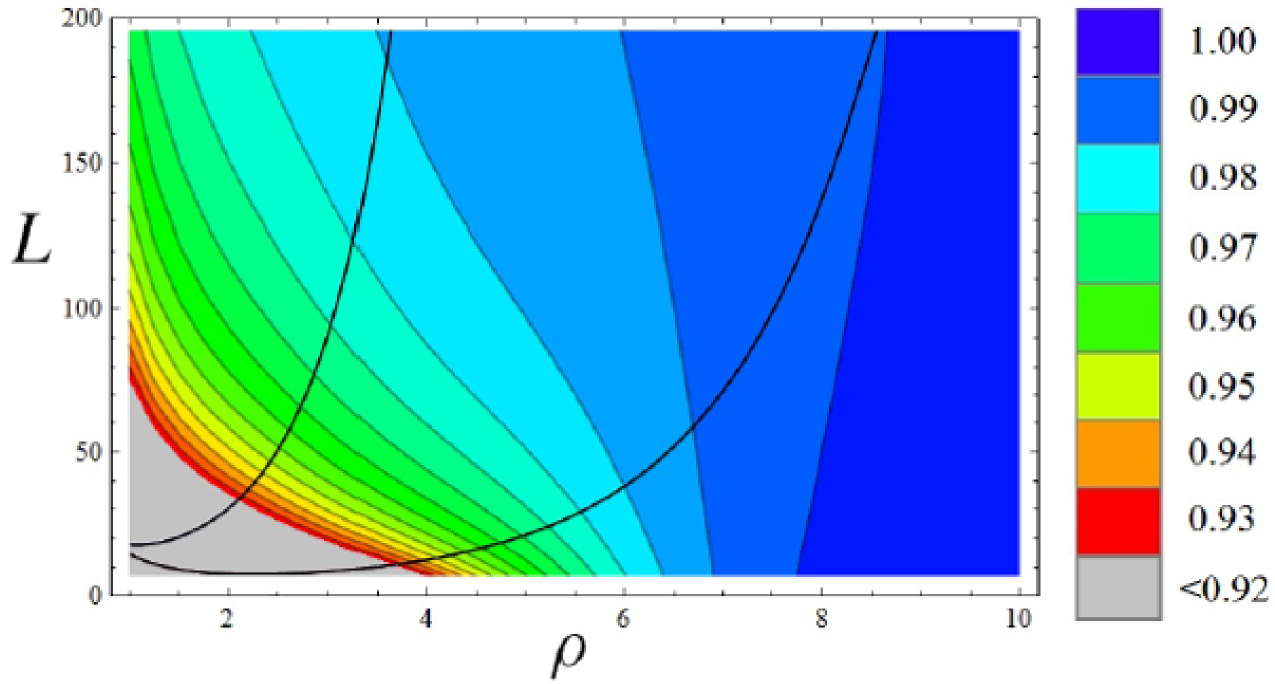
if the integral converges, where k sums over discontinuities.

There are similar results for 3D.

Phase diagram

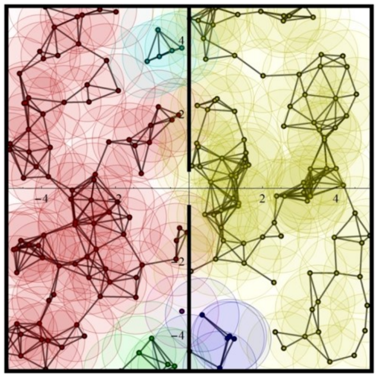
Testing convergence of

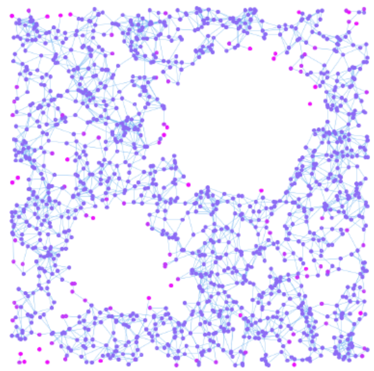
$$\frac{1 - P_{fc}}{\sum_B \cdots}$$

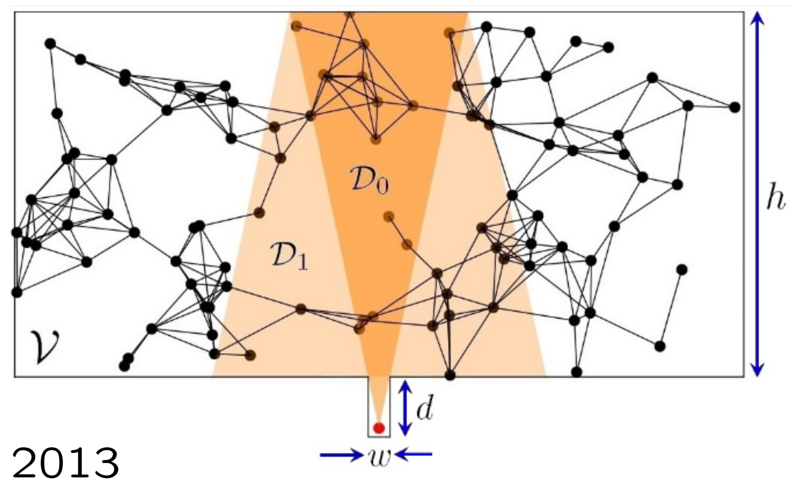


Non-convex geometries

These ideas can be extended to non-convex domains...







Keyholes: OG, CPD and JPC, ISWCS 2013

Obstacles and curved boundaries: A. P. Giles, OG and CPD, arxiv:1502.05440

Reflections: OG, M. Z. Bocus, M. R. Rahman, CPD, JPC,
IEEE Commun Lett 2015

Fractal boundaries: CPD, OG and JPC, ISWCS 2015