Planar growth generates scale free networks.

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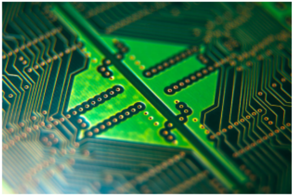
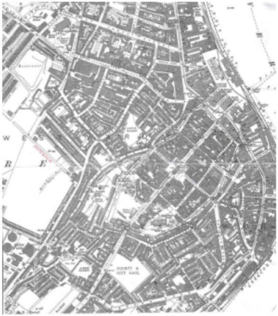
Motivation





Motivation

“What we would really like is some measure of the degree of planarity of a network, a measure the could tell us, for example, that the road network of a country is 99% planar...”
[Newman, 2010]

Motivation

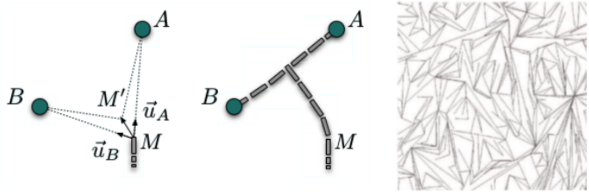


Figure : Planar models have been used to investigate street networks in [Barthélemy and Flammini, 2008] and [Masucci et al., 2009].

Motivation

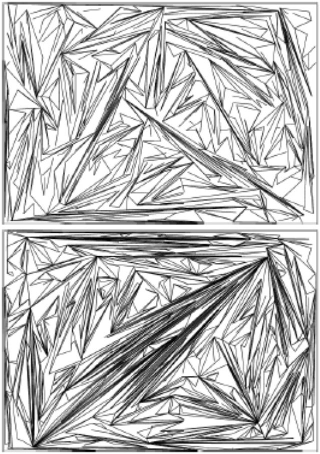


Figure : A planar growth process. [Barthélemy, 2011]

Planar Growth

The model

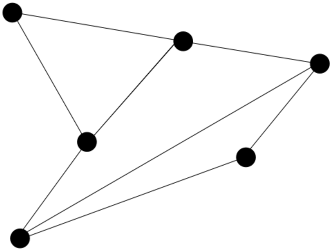


Figure : Begin with a small planar network.

The model

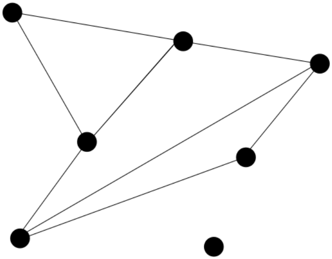


Figure : Grow the network by placing a node on the plane.

The model

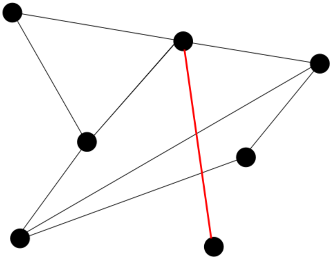


Figure : Reject edges that cross existing ones.

The model

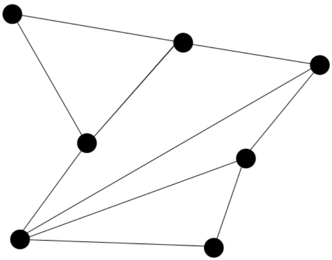


Figure : Add edges until the required degree m is reached.

The model

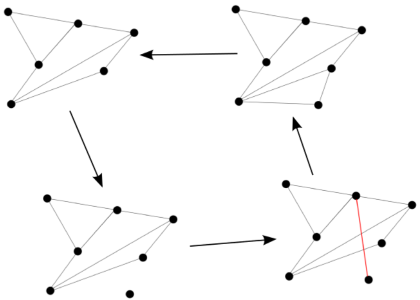


Figure : Repeat until N nodes have been added.

Initial Results

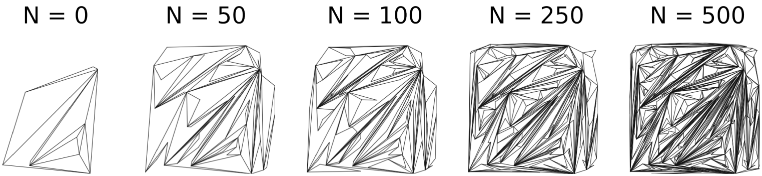


Figure : A PG network at various stages of its growth.

Initial Results

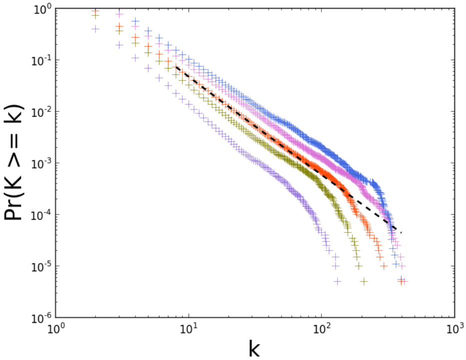


Figure : Degree distribution for networks of order $n = 10^4$ and $m = 1$ (purple), 1.5 (olive), 2 (orange), 2.5 (pink) and 3 (blue).

Reference cases

- PG/planarity
  - planarity is not enforced.
- PG/growth
  - N nodes placed randomly during initialisation.
  - Pairs of nodes chosen at random.

Reference cases

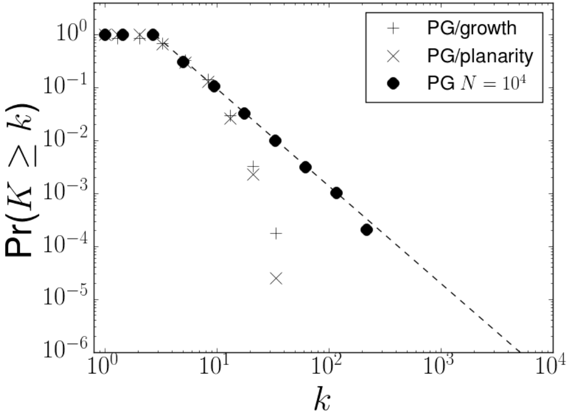


Figure : Degree distributions for each case.

Reference cases

PG PG/planarity PG/growth

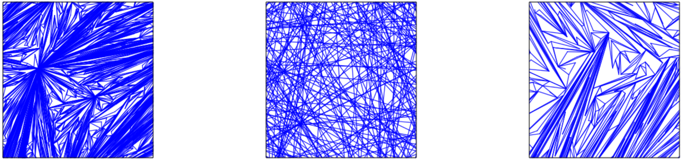


Figure : Embeddings of each case.

Reference cases

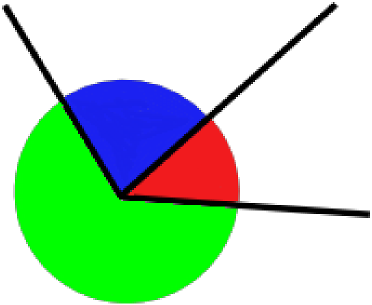


Figure : Measure the angle between each pair.

Reference cases

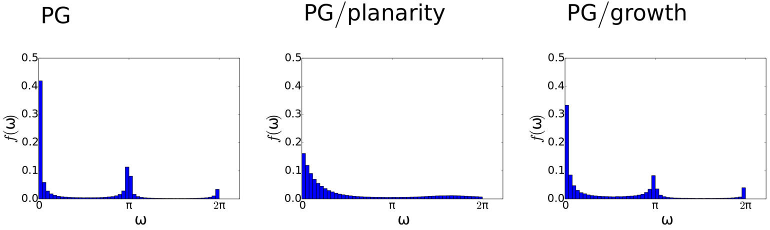


Figure : Angle distributions.

Relaxed planarity

An intermediate model

- New parameter $\chi \in [0, 1]$
- Allow edge crossings with probability χ

An intermediate model

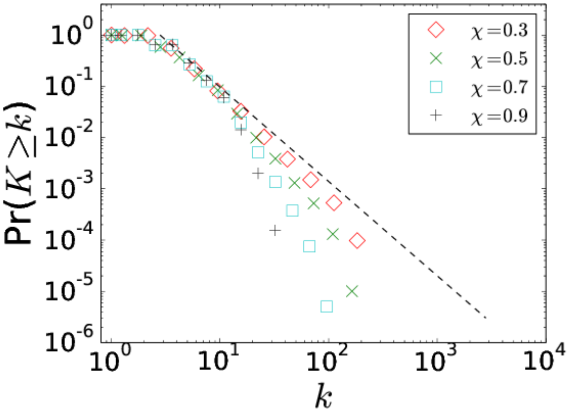


Figure : Degree distributions for each case.

An intermediate model

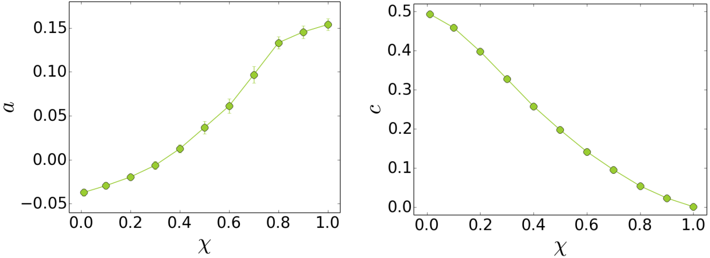


Figure : Assortativity (left) and clustering plotted against χ.

Apollonian Networks

Two existing models

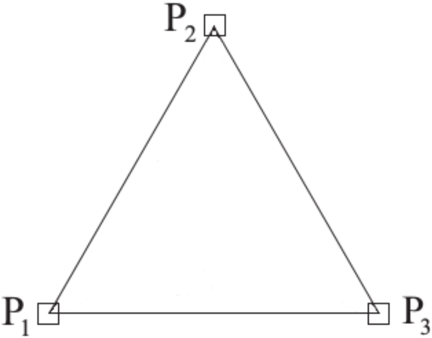


Figure : An Apollonian Network. [Andrade Jr et al., 2005]

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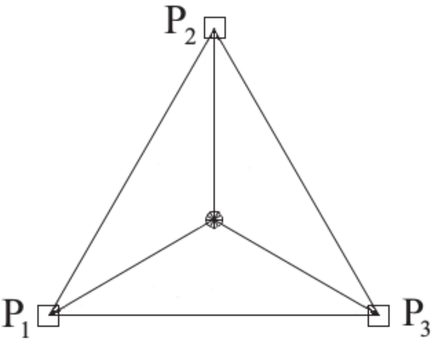

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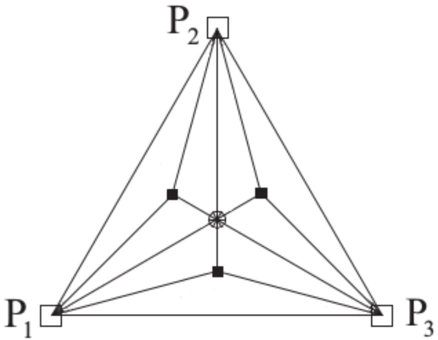

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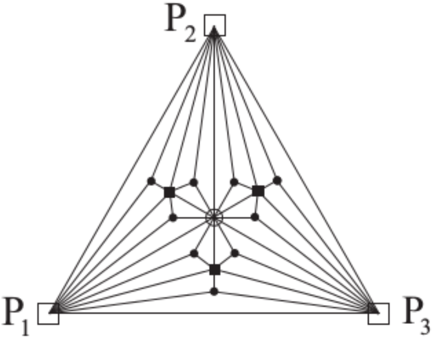

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Random Apollonian Networks

Two existing models

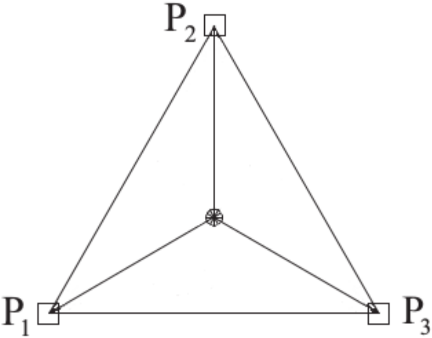
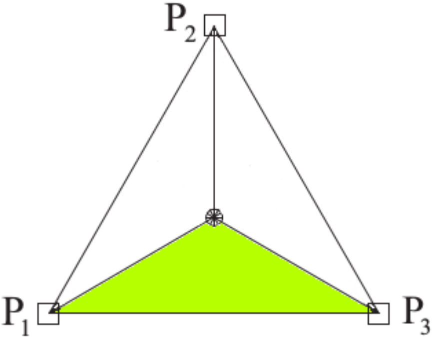


Figure : A Random Apollonian Network. [Zhou et al., 2005]

Two existing models



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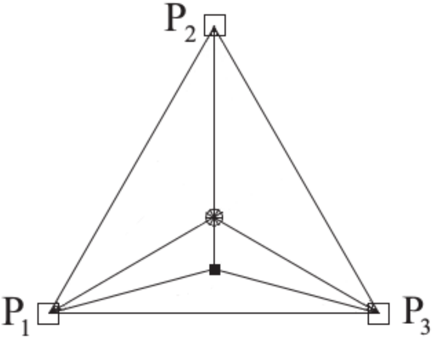
Two existing models



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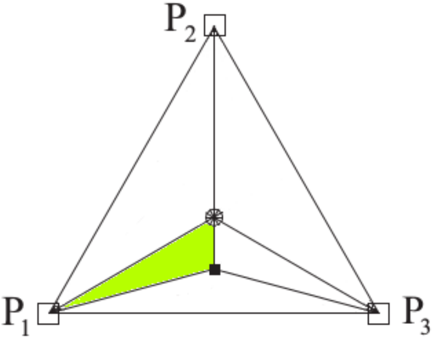
Two existing models



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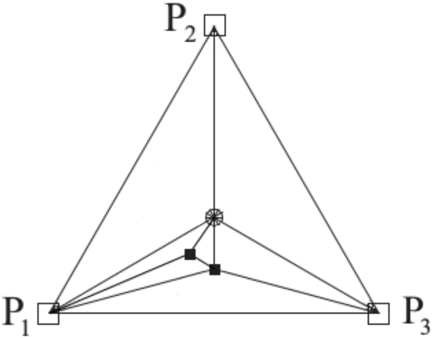

Figure : A Random Apollonian Network. [Zhou et al., 2005]

Apollonian Planar Growth

- Again, select at random from F, the set of all faces.
- Weight the selection by area, i.e.

$$\pi_i = \frac{a_i}{\sum_{j \in F} a_j}$$

- Place a node u.a.r. in the face and connect it to the vertices.

Apollonian Planar Growth

Alternatively, you could just,

- pick, u.a.r., a location on the triangle
- Connect it to the vertices of the face containing the location

In other words, Planar Growth with $m = 3$.

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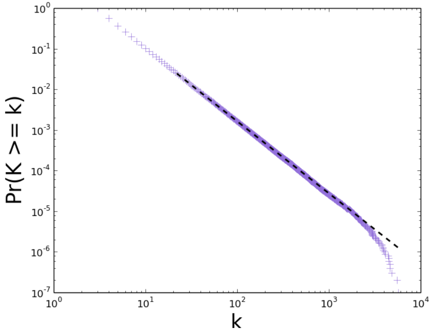

Figure : Degree distribution of a network of order $N = 5 \times 10^5$ grown using Apollonian Planar Growth. Estimated power law exponent is $\alpha = 2.77$.

Variable area weight

Add an exponent to the area weighting formula:

$$\pi_i = \frac{a_i^\beta}{\sum\limits_{j \in F} a_j^\beta}$$

For $0 < \beta \ll 1$ triangle selection is roughly uniform.
For $1 \ll \beta < \infty$ the largest triangle is preferred.

Variable area weight

Table : Estimated exponent of degree distribution for varying area weighting exponent.

β	α
10^{-3}	2.93
10^{0}	2.73
∞	2.76

α - exponent of the degree distribution.
β - area weighting exponent.

Apollonian Planar Growth with Trisection

Variable area weight, $\beta < 1$

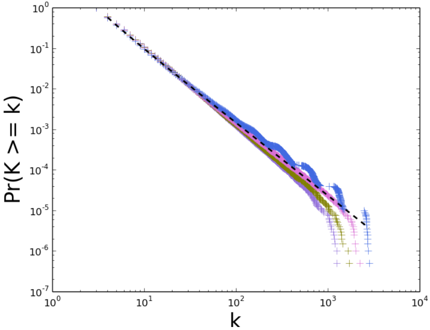

Figure : Degree distribution of networks of order $N = 10^5$ grown using Apollonian Planar Growth with β equal to $10^{-0.5}$, blue; 10^{-1}, pink; $10^{-1.5}$, olive and 10^{-2}, purple.

Variable area weight, $\beta > 1$

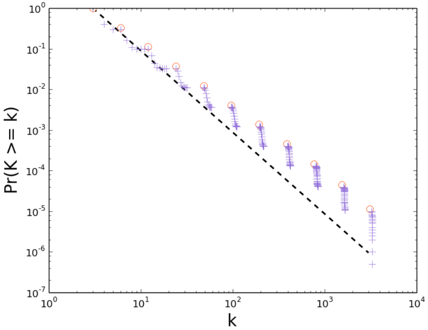

Figure : Degree distribution of a network of order $N = 10^5$ grown using Apollonian Planar Growth with $\beta = 100$. Points in red are the degree distribution of an Apollonian network of the same size.

Variable area weight, $\beta = \infty$

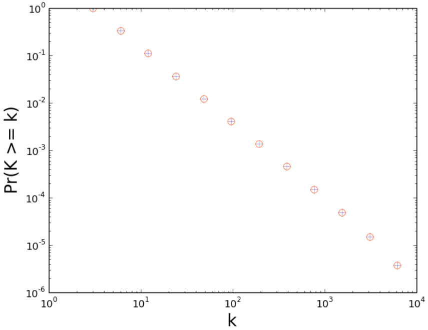

Figure : Degree distribution of a network of order $N = 265720$ grown using Apollonian Planar Growth with $\beta = \infty$. Points in red are the degree distribution of an Apollonian network of the same size.

Summary

- Power law degree distribution simply by maintaining planarity.
- Nodes distributed uniformly in space.
- Generalisation of existing Apollonian networks.

Citations

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