Static traffic assignment with altruistic and selfishly routed vehicles.

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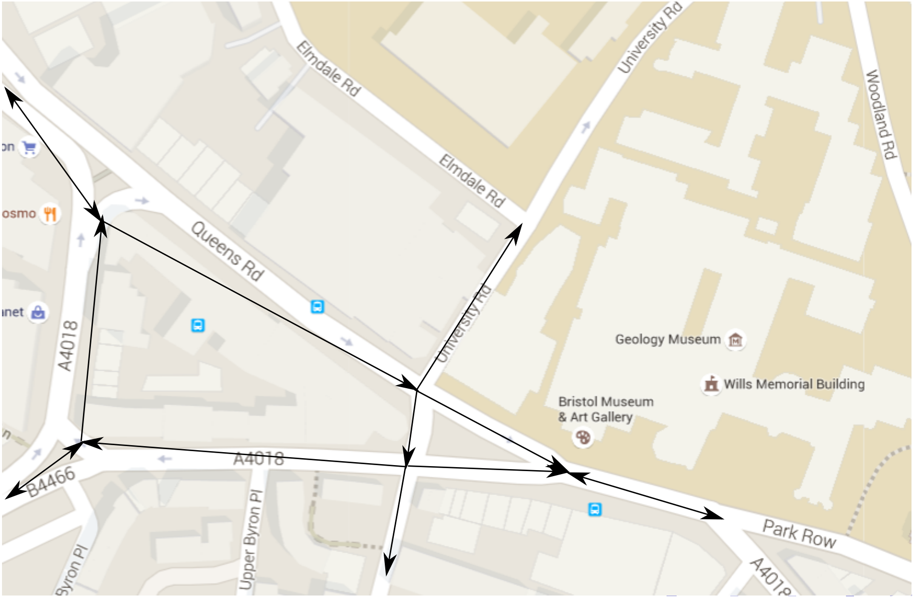
University of BRISTOL

The traffic assignment problem



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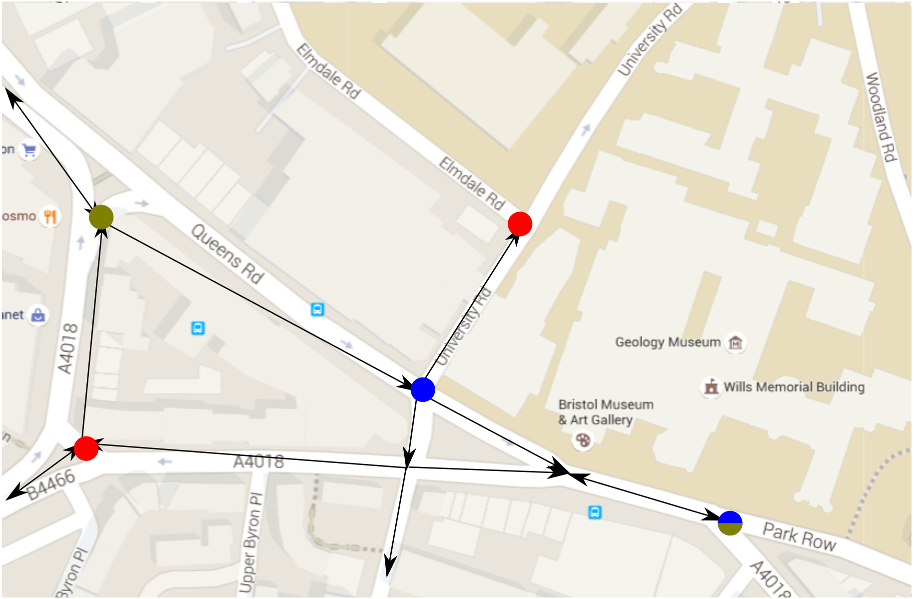

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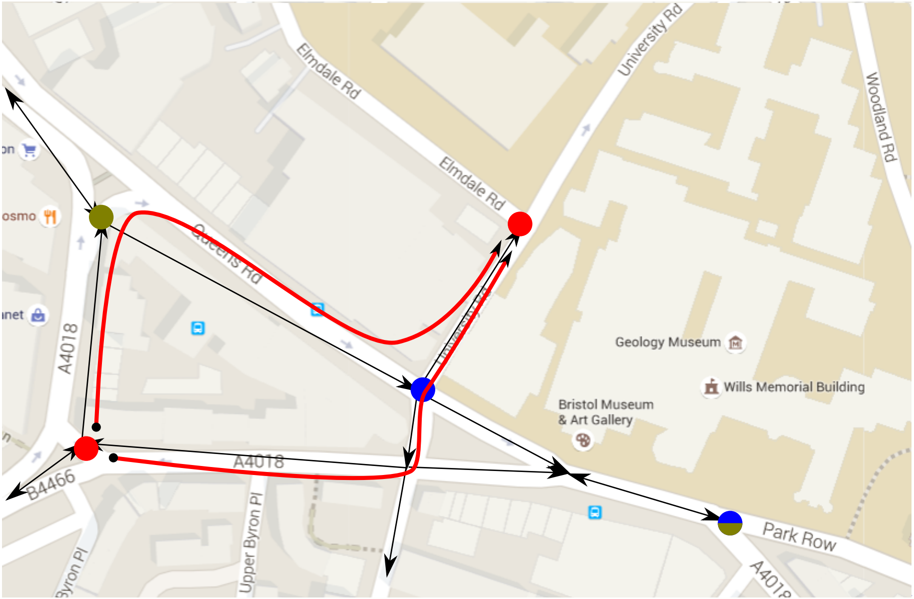

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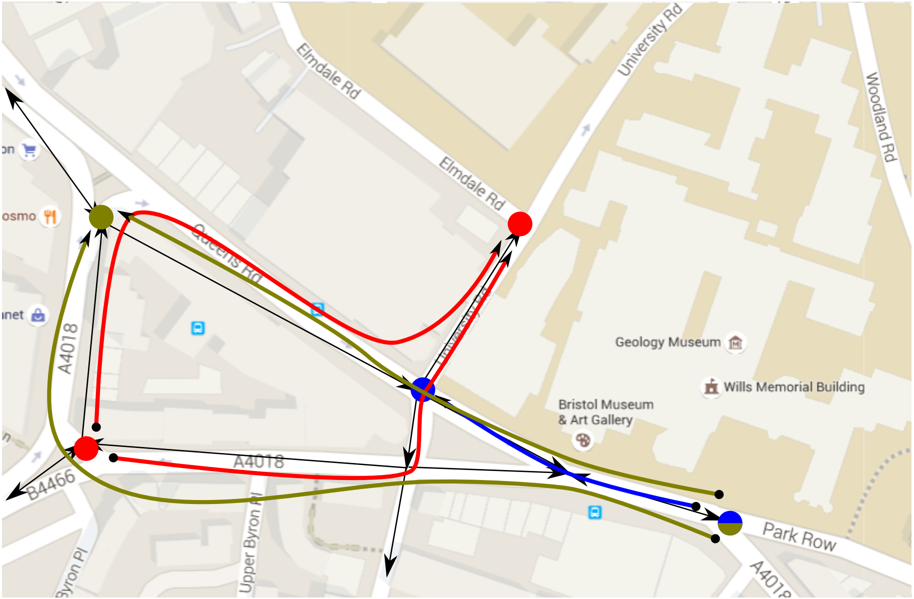

Travel Costs

How much does it cost to use a road?

- Time is money
- Length of road
- Congestion

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Examples of cost functions (x is flow):

- Affine: $f(x) = a + bx$
- BPR function: $f(x) = a + \left(\frac{x}{b}\right)^4$
- Delay functions (queues).

Two different minima

Wardrop's first principle

Journey times on all used routes are equal and less than the free-flow cost of unsused routes.

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Wardrop's first principle

Journey times on all used routes are equal and less than the free-flow cost of unused routes.

Wardrop's second Principle

Average journey time is at minimum / System cost is at minimum.

$$C_T(\mathbf{x}) = \sum_i f_i(x_i) x_i$$

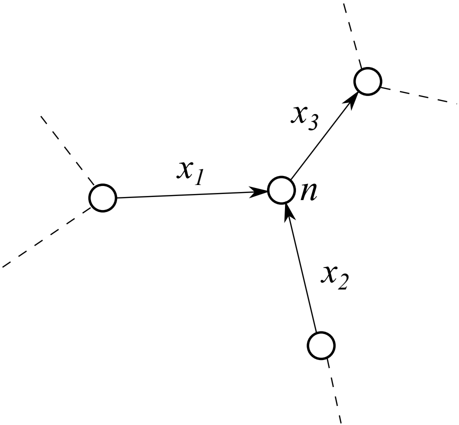
Conservation of flows

- Origin nodes are sources.
- Destination nodes are sinks.
- Flow is conserved:

$$Ax = d$$

A is the **directed** incidence matrix.

$$d_i = \begin{cases} -q, \text{if } i \text{ is origin} \\ q, \text{if } i \text{ is destination} \\ 0, \text{otherwise} \end{cases}$$



$$x_1 + x_2 - x_3 = 0$$

Optimisation formulation

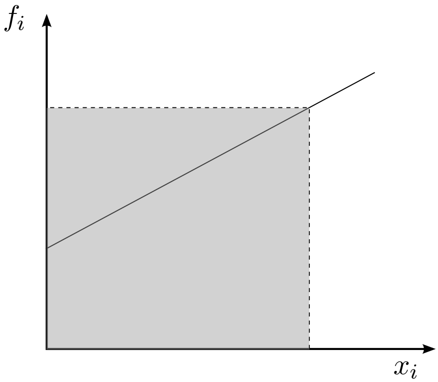
System optimal

$$\begin{aligned} \underset{x}{\text{Minimise}} \quad & \sum_i f_i(x_i) x_i \\ \text{Subject to} \quad & Ax = d \\ & x \succeq 0 \end{aligned}$$

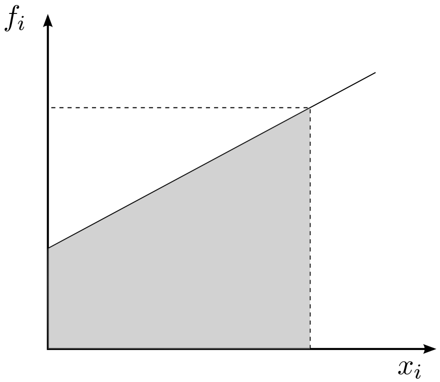
User equilibrium

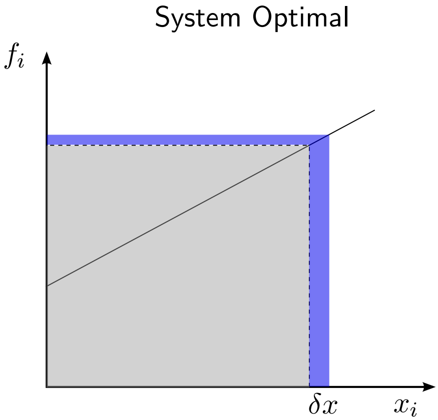
$$\begin{aligned} \underset{x}{\text{Minimise}} \quad & \sum_i \int_0^{x_i} f_i(s) ds \\ \text{Subject to} \quad & Ax = d \\ & x \succeq 0 \end{aligned}$$

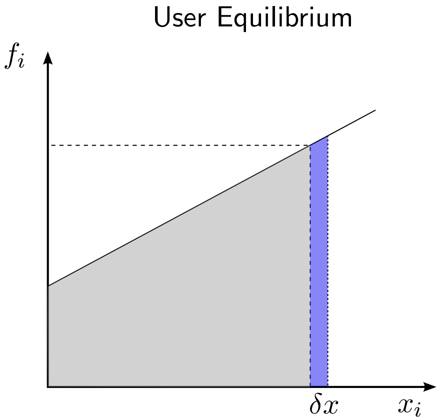
System Optimal



User Equilibrium




System Optimal
f_i
δx
x_i

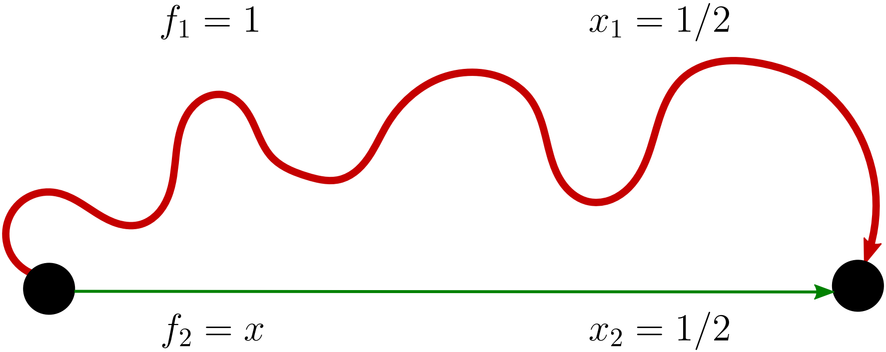

User Equilibrium
f_i
δx
x_i

Pigou's example

$f_1 = 1$

$f_2 = x$

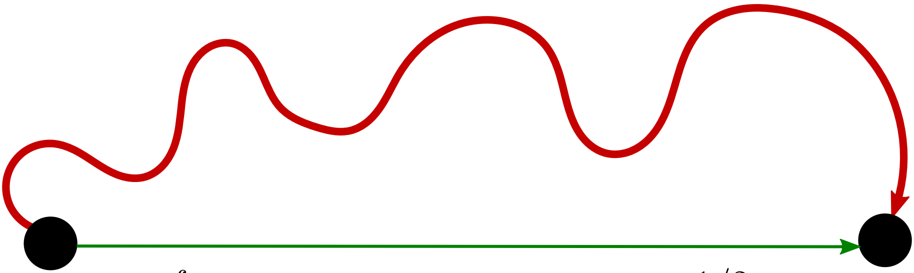
$x_1 + x_2 = 1$



$$C_T = 1 \cdot (1/2) + 1/2 \cdot (1/2)$$

System Optimal

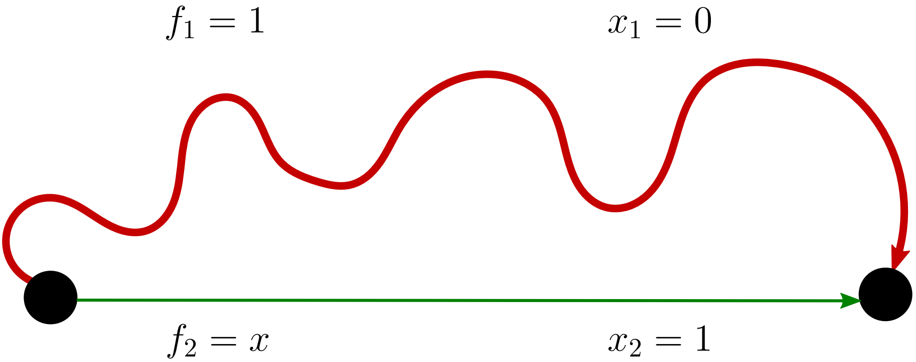
$f_1 = 1$ $\qquad$ $x_1 = 1/2$



$f_2 = x$ $\qquad$ $x_2 = 1/2$

$$C_T = 3/4$$

User equilibrium



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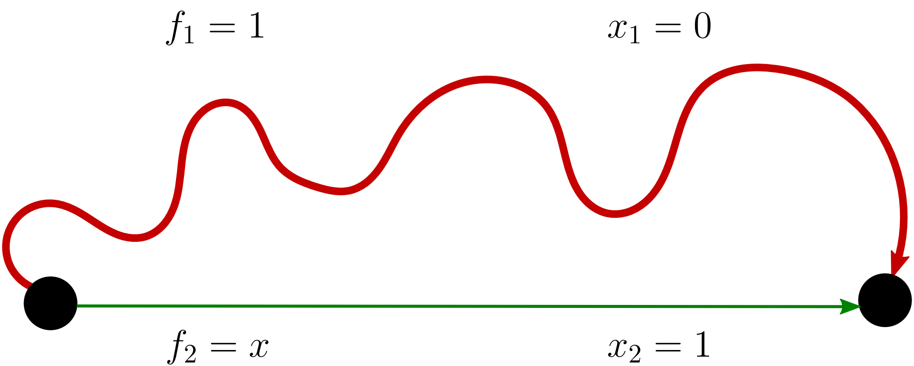


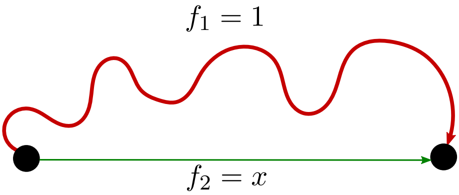
$$C_T = 1$$

Price of anarchy

- Ratio of selfish routing to system optimal performance.

Price of anarchy

$$\mathsf{PoA} = \frac{C_T(\mathbf{x}_{\mathsf{UE}})}{C_T(\mathbf{x}_{\mathsf{SO}})}$$



$$\mathsf{PoA} = \frac{4}{3}$$

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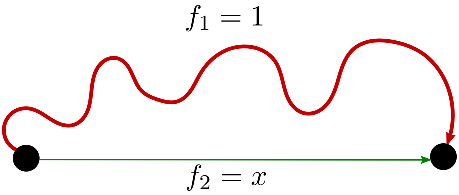


$$\text{PoA} = \frac{4}{3}$$

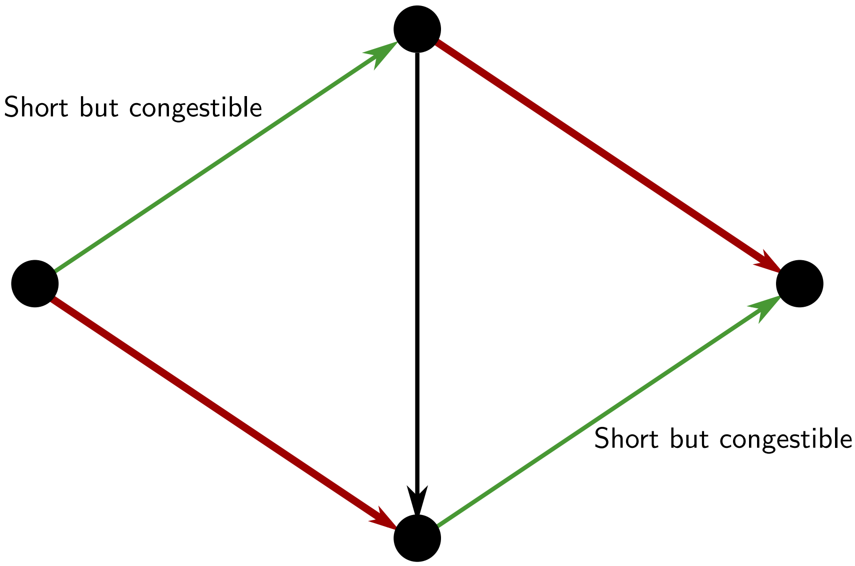
Worst case is achieved!

Price of anarchy

- Depends on available routes.
- It depends on the cost functions.
- Simple networks can be worst-case.
- Is tied to switches.
  (Steven O'Hare PhD thesis)

So where does network structure come into play?

Braess Network



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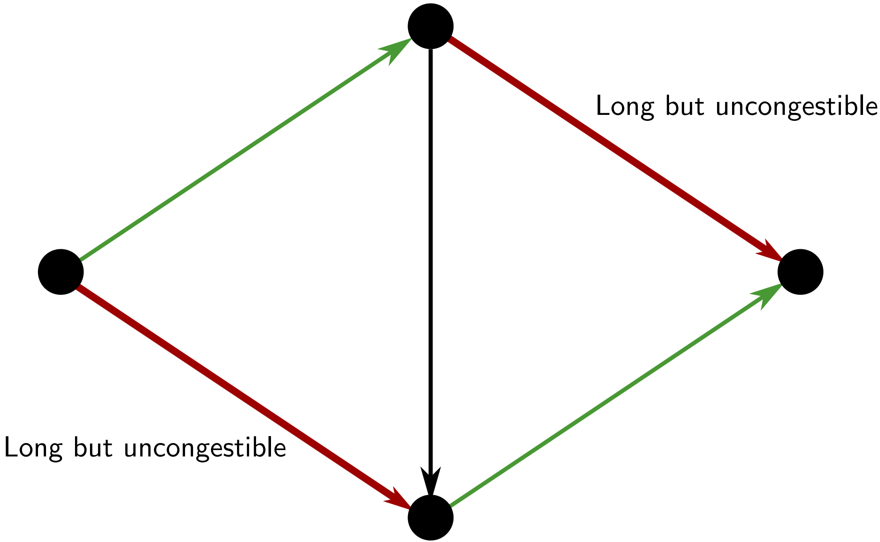
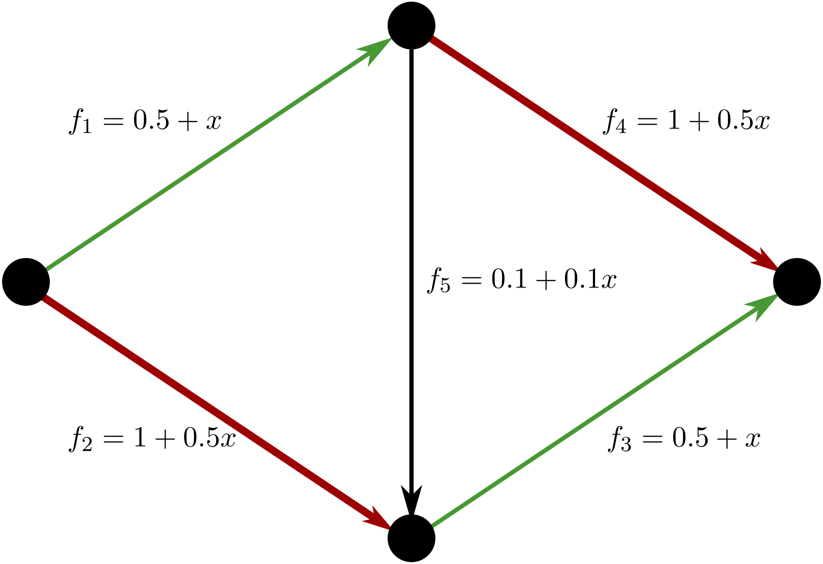
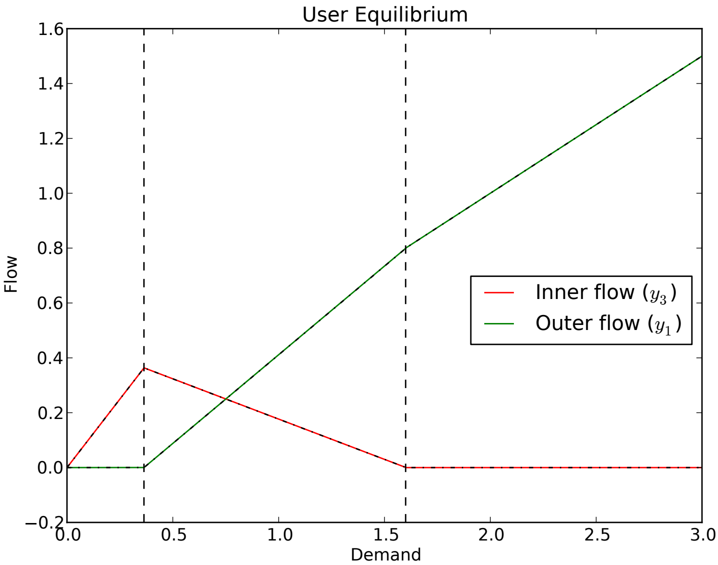


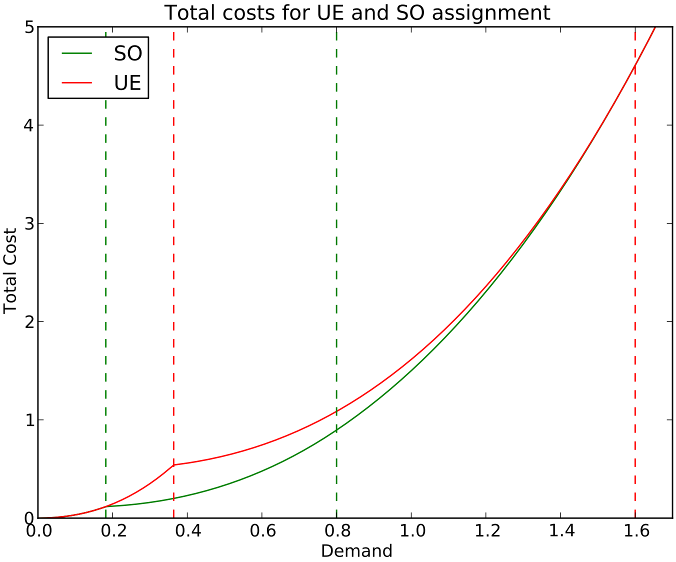
Braess Network



Flow (Braess)

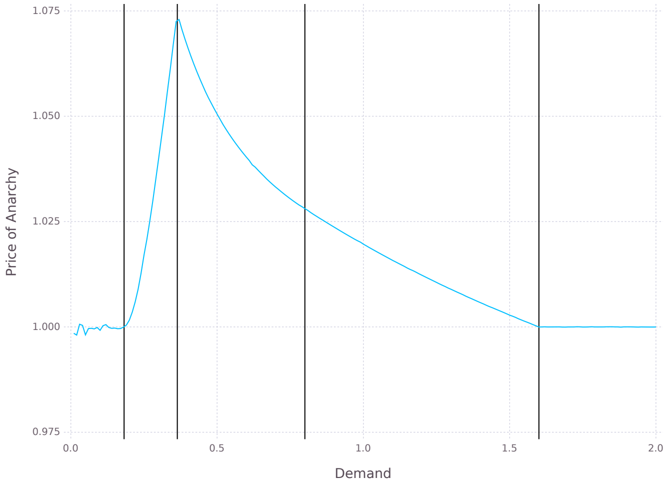

User Equilibrium
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Flow
Demand
Inner flow (y_3)
Outer flow (y_1)

Cost (Braess)

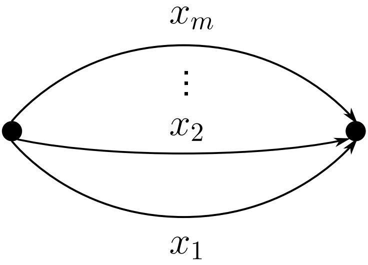

Total costs for UE and SO assignment
SO
UE
Total Cost
Demand
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6

Price of anarchy and switches (Braess)

Effects of switching



Parallel links (non-interacting routes)



- Activation demands:

$$q_i = q_{i-1} + \Delta a_{i-1} \sum_{j=1}^{i-1} \frac{1}{b_j}$$

- Flows:

$$x_i(q) = (q - q_\ell)\frac{1}{\sum_{j=1}^{\ell} \frac{b_i}{b_j}} + \sum_{k=i}^{\ell-1} \Delta q_k \frac{1}{\sum_{j=1}^{k} \frac{b_i}{b_j}}$$

The importance of switching

Interacting routes give us:

- Non-monotonic flows.
- Switches in *active link set*.
- PoA changes due to switch lag.

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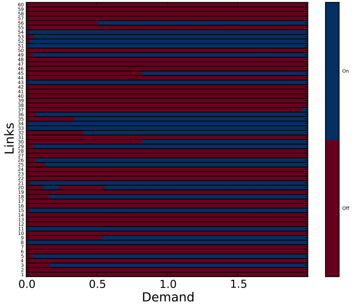
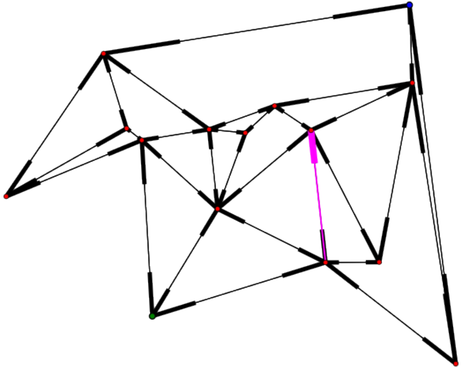
Driving network to Optimal $\Longleftrightarrow$ Inducing switches at SO levels

More complex switching





Altruistic vehicles

Introduce some altruistic vehicles:

- $d^a = \varepsilon d$
- $d^s = (1 - \varepsilon) d$

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Modification of cost functions:

- $f_i(x^a + x^s) = (a_i + x^a b_i) + b_i x^s = \hat{a}_i + b_i x^s$

A Stackelberg game

- 2-player game
- Leader: Network manager (Altruistic vehicles)
- Follower(s): selfish cohort of vehicles

Bilevel optimisation formulation

$$\underset{x^a}{\text{Minimise}} \quad C_T(x^a + x^s)$$

Subject to

$$\underset{x^s}{\text{Minimise}} \quad \sum_i \int_0^{x_i} f_i(s)ds$$

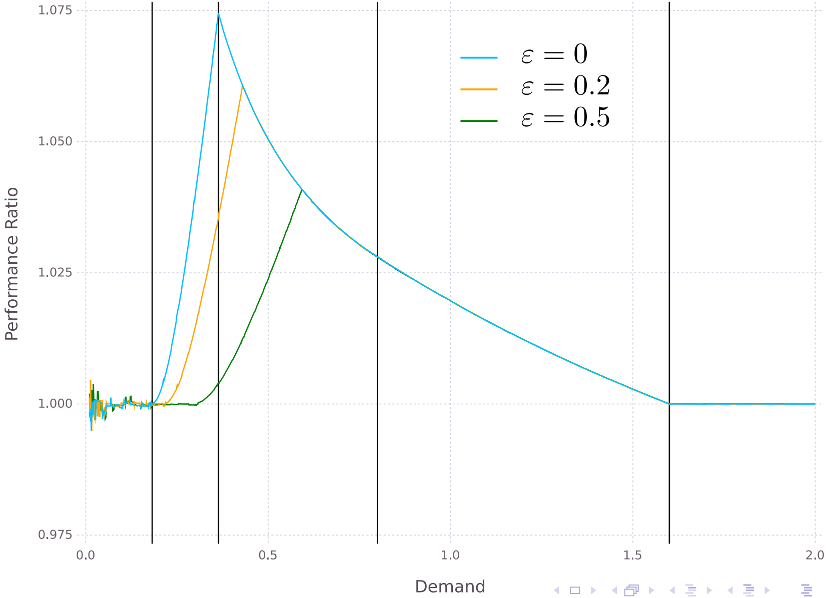
$$\begin{aligned} \text{Subject to} \quad & Ax^a = (1-\epsilon)d \\ & Ax^s = \epsilon d \\ & x^a \succeq 0 \\ & x^s \succeq 0 \end{aligned}$$

Objective function

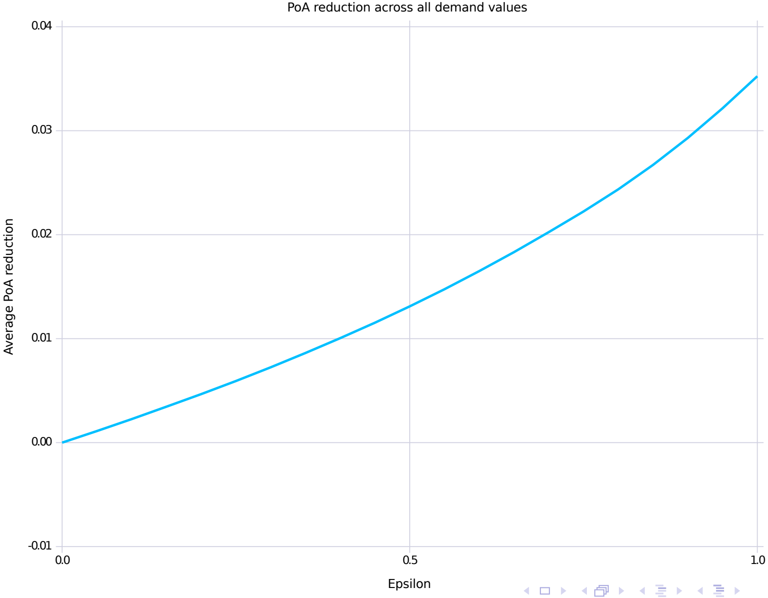
What should the *altruistic* vehicles optimise?

- Total system cost.
- SO amongst themselves.
- What incentives are there?

Back to Braess network



PoA reduction



Conclusions

- Switching is key.
- Switches can be induced early using mixed assignment.
- Can only modify costs of roads with altruistic flow (not directly controlled)
- For small values of demand PoA can be held at 1.
- Optimal percentage of altruistic vehicles has to be done externally.
- Network analysis (centrality etc...) has to be done after assignment

Thank You

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