Exploiting symmetries in network analysis

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University of Sheffield

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A brief introduction...

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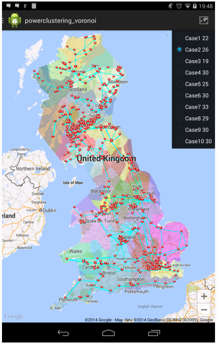

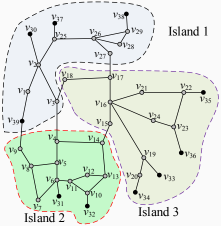

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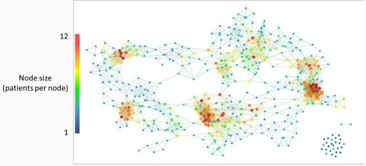

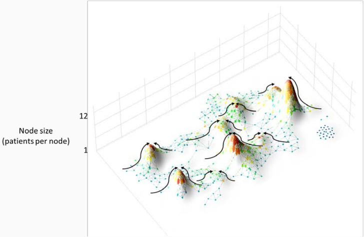

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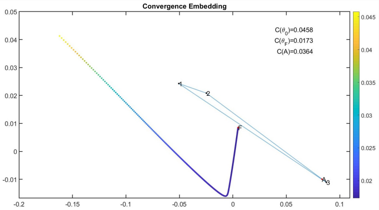

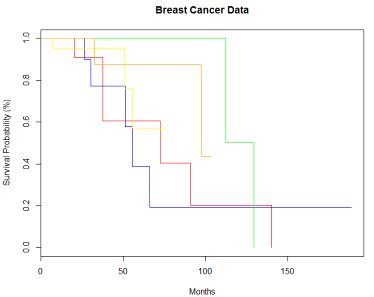

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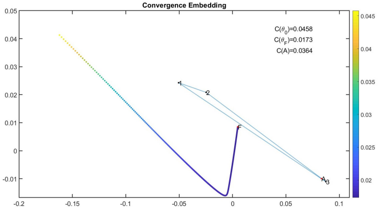
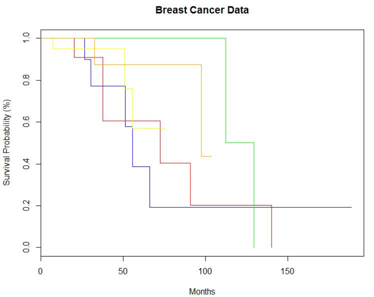
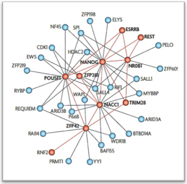
  - network symmetry. . .

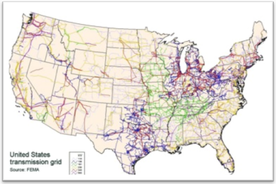


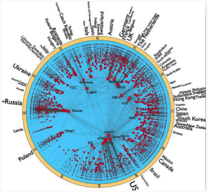


Real-world complex networks

- Network models of real-world complex systems (biological, social, technological, . . .)



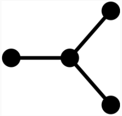




- Successful approach
  - simple & versatile
  - reveals structural and dynamical properties
  - common properties/principles e.g. symmetry

Network symmetry

- Network symmetries identify structurally equivalent nodes (and links)



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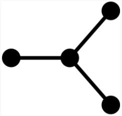

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(Here $\mathcal{G} = (V, E)$ graph with vertex set V and edge set $E \subseteq V \times V$)

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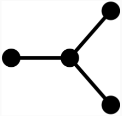

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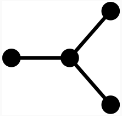

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- They may arise from replicative growth processes, evolution from basic principles or functional optimisation

Are real-world networks symmetric?

[MacArthur, Sanchez-Garcia, Anderson *Symmetry in Complex Networks* **Discrete Appl. Math.** (2008)]

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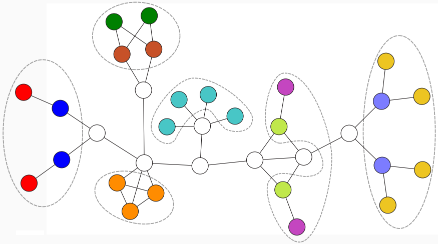

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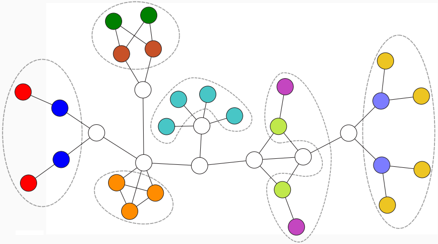

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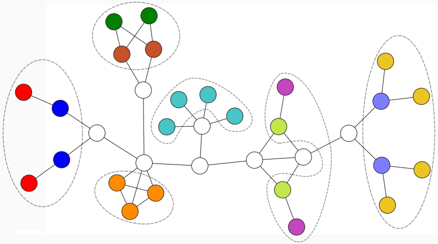

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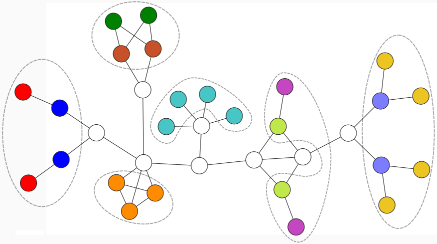

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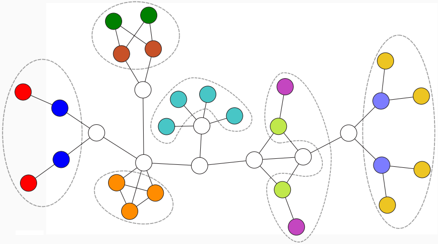

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- Non-basic symmetric motifs typically branched trees

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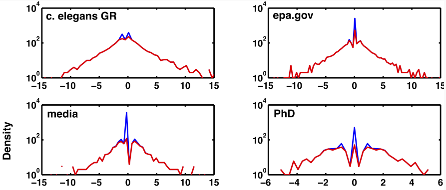
Redundant spectrum

- Symmetry explains most of the discrete part of the spectrum ('peaks' in the spectral density) of the adjacency matrix



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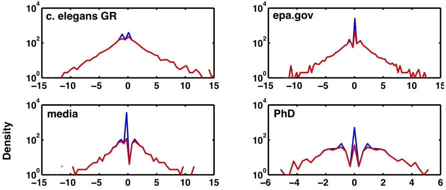

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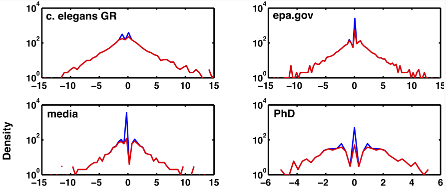

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- Most symmetric motifs are basic and we can predict their contribution to the discrete spectrum e.g.

$$\mathrm{RSpec}_1 = \{-1, 0\} \qquad \mathrm{RSpec}_2 = \{-2, -\varphi, -1, 0, \varphi - 1, 1\}$$

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Graph automorphisms

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- In particular, automorphisms generate high-multiplicity eigenvalues: $Av = \lambda v$ implies $APv = PAv = \lambda Pv$.

Network measures

Crucial observation

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- A pairwise network measure $F\colon V \times V \to \mathbf{R}$ is *structural* if $F(i,j) = F(\sigma(i),\sigma(j))$ for all $\sigma \in \mathrm{Aut}(\mathcal{G})$ and all $i,j \in V$.
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Note Our results can be adapted to the presence of vertex/edge labels by restricting to automorphism preserving the additional structure

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[Sanchez-Garcia *Exploiting symmetries in network analysis* arxiv preprint 1803.06915]

Full and sparse network measures

Our results apply to arbitrary network measures, however some will be more useful for either *full* or *sparse* measures.

full measure $F(i,j) \neq 0$ for most $i,j \in V$ (e.g. graph metric)

sparse measure $F(i,j) = 0$ if $a_{ij} = 0$ for most $i,j \in V$ (e.g. graph Laplacian)

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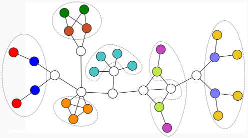

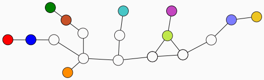

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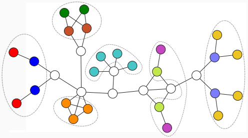

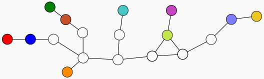

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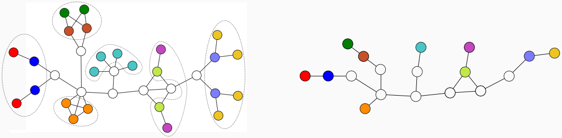

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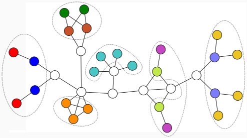
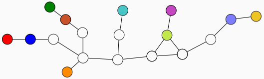
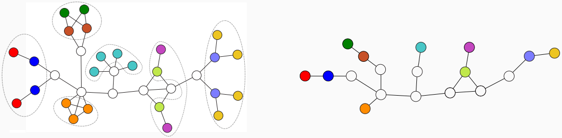
- Matrix equation: $Q(A) = \Lambda^{-1} S^T A S$ where S is the $n \times m$ *characteristic matrix* of the partition ($[S]_{ik} = 1$ if $i \in V_k$) and $\Lambda = \mathrm{diag}(n_1, \ldots, n_m)$, $n_k = |V_k|$.





Network quotient

- Formal procedure to eliminate redundancies/symmetries.
- Let $\mathcal{G} = (V, E)$ with $n \times n$ adjacency matrix $A = (a_{ij})$ and orbit partition $V = V_1 \cup \ldots \cup V_m$.
- The *quotient graph* $\mathcal{Q}$ is the graph with adjacency matrix $Q(A) = (b_{kl})$ given by $b_{kl} = \frac{1}{|V_k|} \sum_{i \in V_k, j \in V_l} a_{ij}$ (average connectivity from a vertex in V_k to vertices in V_l).
- Matrix equation: $Q(A) = \Lambda^{-1} S^T A S$ where S is the $n \times m$ *characteristic matrix* of the partition ($[S]_{ik} = 1$ if $i \in V_k$) and $\Lambda = \mathrm{diag}(n_1, \ldots, n_m)$, $n_k = |V_k|$.
- The graph $\mathcal{Q}$ is directed, weighted with loops, but spectrally equivalent to an undirected graph, if $\mathcal{G}$ undirected too.



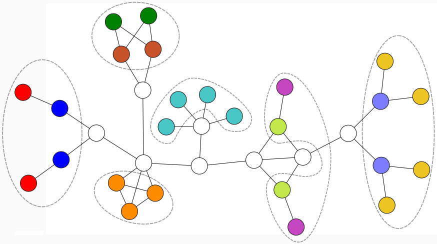
(Left) Toy network (Right) Quotient skeleton (no loops, edge directions or weights)

Internal and external edges

Internal edges between vertices in the same symmetric motif

External edges between vertices in different symmetric motifs

(fixed points treated as trivial symmetric motifs of size 1)



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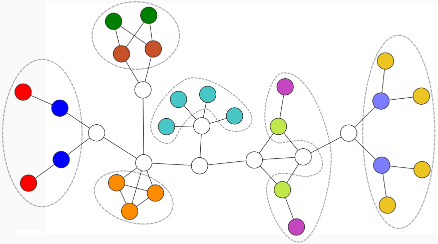

- The vast majority of edges in $F(\mathcal{G})$ are external (typically over 90% in the sparse, and 99.99% in the full, case)

Symmetry compression

- $F(\mathcal{G})$ inherits all the symmetries of $\mathcal{G}$ as redundancies, namely repeated values $F(\sigma(i), \sigma(j)) = F(i, j)$.
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- Compression ratios between $m_{\mathcal{Q}}/m_{\mathcal{G}}$ (sparse) and $(n_{\mathcal{Q}}/n_{\mathcal{G}})^2$ (full).

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Table 2. Symmetry in some real-world networks.

Name	$n_{\mathcal{G}}$	$m_{\mathcal{G}}$	gen	t_1	t_2	sm	bsm	mv	$\tilde{n}_{\mathcal{Q}}$
HumanDisease	1,419	2,738	713	0.00	0.16	272	96.0	71.0	48.3
Yeast	1,647	2,736	380	0.00	0.01	149	99.3	33.3	76.3
OpenFlights	3,397	19,230	732	0.00	0.11	321	93.5	32.4	77.3
USPowerGrid	4,941	6,594	414	0.00	0.09	302	97.4	16.7	90.2
HumanPPI	9,270	36,918	972	0.00	0.12	437	100	15.3	89.5
Astro-Ph	17,903	196,972	3,232	0.01	0.21	1,682	99.4	27.5	81.9
InternetAS	34,761	107,720	15,587	0.03	0.29	3,189	99.9	54.3	55.0
WordNet	145,145	656,230	52,152	0.18	0.62	28,456	92.0	60.0	60.1
Amazon	334,863	925,872	32,098	0.20	0.39	23,302	99.8	16.8	90.3
Actors	374,511	15,014,839	182,803	0.95	1.38	36,703	99.9	58.6	51.2
InternetAS-skitter	1,694,616	11,094,209	319,738	1.71	4.17	84,675	99.1	19.7	85.4
CaliforniaRoads	1,957,027	2,760,388	36,430	0.47	0.16	35,210	98.8	4.0	97.9
LiveJournal	5,189,808	48,687,945	410,575	8.02	3.59	245,211	99.9	12.7	92.1

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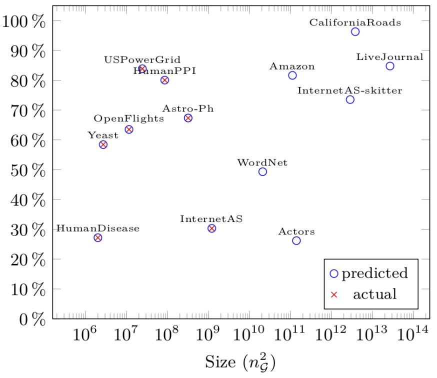

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- Pseudocode in the paper, and MATLAB code available in Bitbucket

Computational reduction

- (**Orbit reduction**) $F(i,j) = F(\sigma(i), \sigma(j))$ means we only need to evaluate F on $m_{\mathcal{Q}} < m_{\mathcal{G}}$ (sparse) or $n_{\mathcal{Q}}^2 \ll n_{\mathcal{G}}^2$ (full) vertex pairs

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- In practice we find different 'degrees' of recoverability: we call F *partially*/*average*/*fully* quotient recoverable if the external/external & averaged internal/external & internal edges of $F(\mathcal{G})$ can be obtained from $F(\mathcal{Q})$

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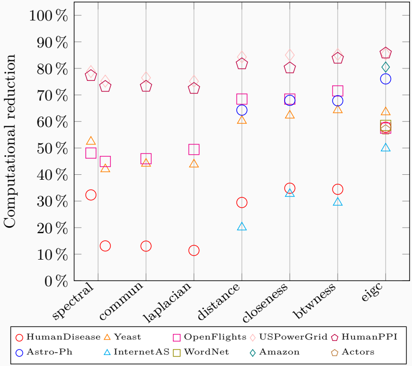

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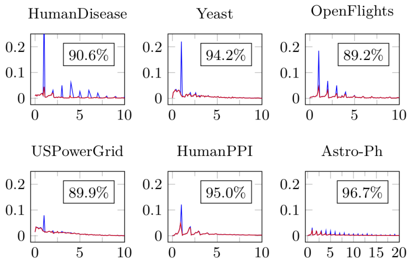

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- Example: for the graph Laplacian and 1-orbit BSMs, we obtain $\mathbb{Z}^+$

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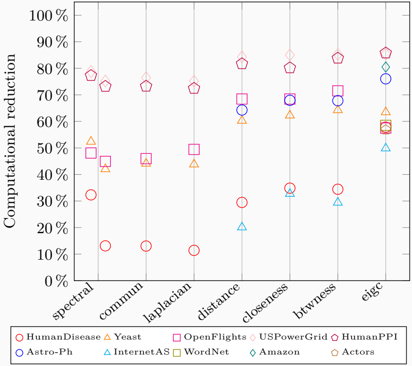

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Applications (examples)

- We studied symmetry compression, computational reduction, and the redundant spectrum of several well-known network measures:
  - communicability ($f(A) = \sum_{n=0}^{\infty} a_k A^k$)
  - shortest path distance
  - resistance metric (equivalent to $L^\dagger$)
  - adjacency matrix (supersedes [MSA08] and [MS09])
  - graph Laplacian

Full details in the preprint Sanchez-Garcia *Exploiting symmetries in network analysis* arxiv/1803.06915

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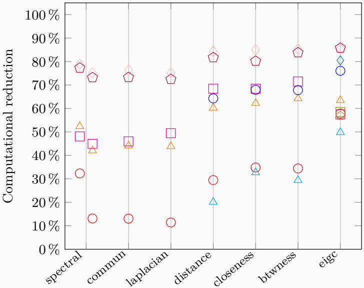

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- Quotient reduction often holds (e.g. degree, eccentricity, closeness and eigenvector centrality)
- Often vertex measures arise from pairwise measures
  e.g. $G(i) = F(i,i)$ or $G(i) = \frac{1}{n}\sum_j F(i,j)$.

Conclusions

For arbitrary (pairwise) structural network measures we have shown:

- a general framework to describe, manipulate and quantify the inherited symmetry and redundancy on an arbitrary network measure
- symmetry compression algorithms with average or lossless compression
- how to use the quotient network for computational reduction
- the contribution of symmetry to the discrete part of the spectrum
- how to extend the results on compression and computational reduction to arbitrary vertex-based measures
- illustrated our methods in several pairwise and vertex-based measures in empirical networks up to several million nodes

References

[MSA08] MacArthur, Sanchez-Garcia, Anderson *Symmetry in Complex Networks* **Discrete Appl. Math.** (2008)

[MS09] MacArthur, Sanchez-Garcia *Spectral characteristics of network redundancy* **Phys. Rev. E** (2009)

[preprint] Sanchez-Garcia *Exploiting symmetries in network analysis* arxiv preprint 1803.06915

[BitBucket] `https://bitbucket.org/rubenjsanchezgarcia/networksymmetry/`

Questions?